Scaling of Sea Ice Fracture—Part II: Horizontal Load From Moving Ice

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Continuing the analysis of fracture size effect in Part I, which was focused on the maximum force in vertical penetration of ice, Part II tackles the problem maximum force that can be applied by a moving ice plate on an obstacle presented by a fixed structure. Based on an asymptotic approach, approximate solutions for are obtained for the size effects of ice thickness, effective structure diameter and, in the case of a finite ice floe, the size of the floe. [DOI: 10.1115/1.1429933]

1 Introduction

After analyzing in Part I the vertical penetration problem, we will examine in the present Part II another fundamental problem of large-scale fracture of sea ice—the maximum force P that can be exerted by a moving ice plate of thickness h on an obstacle presented by a fixed structure of effective diameter d (imagined as a cylinder). Similar simplifications will be made and the approach of asymptotic matching will again be followed. All the definition and notations from Part I will be retained. Several possible mechanisms of breakup will be considered.

Stress analysis and fracture of floating ice plates subjected to a horizontal load has been studied by Ashton, Atkins, Goldstein and Osipenko, Lavrov, Palmer et al., Ponter and Slepyan, and others ([1–7]). These investigators used dimensional analysis to determine the scaling laws of linear elastic fracture mechanics (LEFM) and of strength theory. They did not consider cohesive fracture and did not attempt to bridge these two theories to describe the size effect transition from one to the other. Characterizing this transition is the main objective of what follows.

For the horizontal load, it is convenient to define the nominal strength as the average stress on the cross-section area hd of the structure facing the moving ice plate, i.e.,

$$\sigma_N = P/hd. \tag{1}$$

2 Global Failure due to Buckling of Ice Plate

Cylindrical buckling, in which the deflection surface is a translationary surface, can occur only if the ice plate is moving against a very long wall ($d\to\infty$). In this case the plate behaves as a beam on elastic foundation, which is a one-dimensional problem, and the critical compressive normal force per unit width of the plate is known to be (e.g., [8]) $N_{cr}=\kappa_0\sqrt{\rho D}$ where coefficient κ_0 depends on the boundary conditions. Its minimum value occurs for a semi-infinite plate with a straight infinite free edge and is $\kappa_0=1$.

If the obstacle, such as the legs of an oil drilling platform, has a finite dimension d in the transverse direction, the buckling mode is two-dimensional and more complicated. In any case, however, dimensional analysis ([9,10]) suffices to determine the form of the buckling formula and the scaling ([3,7]).

There are five variables in the problem, P_{cr}, E', ρ, h, d, and the solution must have the form $F(P_{cr},E',\rho,h,d)=0$, where $E'=E/(1-\nu^2)$; and P_{cr} is the critical force exerted by the resisting structure on the moving ice plate (Fig. 1(*a*)). There are, however, only two independent physical dimensions in the problem, namely the length and the force. Therefore, according to Buckingham's Π theorem of dimensional analysis ([9–11]), the solution must be expressible in terms of $5-2$, i.e., 3 dimensionless parameters. They may be taken as $P_{cr}/E'hd$, $\sqrt{\rho D}/E'h$ and d/h. Because the buckling is linearly elastic, $P_{cr}/E'hd$ must be proportional to $\sqrt{\rho E'}/E'h$ and d/h. Denoting

$$\sigma_{N_{cr}}=P_{cr}/hd \tag{2}$$

which represents the nominal buckling strength (or the average critical stress applied by the face of the resisting structures on the moving ice plate), and noting that $D=E'h^3/12$ with $E'=E/(1-\nu^2)$, we conclude that the buckling solution must have the form

$$\sigma_{N_{cr}}=\kappa(d/h)\sqrt{\rho E'}\sqrt{h} \tag{3}$$

where κ is a dimensionless parameter depending on the relative diameter of the structure, d/h, as well as on the boundary conditions. For $d/h\to\infty$ (i.e., an infinite wall), this must reduce [8] to the critical stress for an infinite beam on elastic foundation loaded at the free end (vertically sliding end). Therefore, $\kappa(0)/\sqrt{12}=1$ or $\kappa(0)=2\sqrt{3}$, which represents the smallest possible value of κ for any d/h. This fact becomes obvious by imagining a strip of width d in the direction of movement to be separated from the rest of the ice plate; for that strip $\kappa_0=1$ (if the ice in contact with the structure is free to slide vertically, Fig. 1(*a*)), and re-attaching the rest of the plate cannot but increase the critical load.

An interesting property of (3) is that, for geometrically by similar structures (constant d/h), $\sigma_{N_{cr}}$ increases, rather than decreases, with ice thickness h. So there is a *reverse* size effect. Consequently, the buckling of the ice plate can be the mechanism of failure only when the plate is sufficiently thin. The reason for the reverse size effect is that the buckling wavelength (the distance between the inflexion points of the deflection profile), which is $L_{cr}=\pi(D/\rho)^{1/4}$ (as follows from dimensional analysis, or from nondimensionalization of the differential equation of plate buckling), is not proportional to h; rather

$$L_{cr}/h\propto h^{-1/4}, \tag{4}$$

i.e., L_{cr} decreases with h. This contrasts with the structural buckling problems of columns, frames, and plates, in which L_{cr} is proportional to the structure size. (Despite the analogy between axisymmetric buckling of an axially compressed cylindrical shell and a floating ice, no size effect occurs for the shell because, unlike ρ_w, the equivalent foundation modulus of the shell scales with h.)

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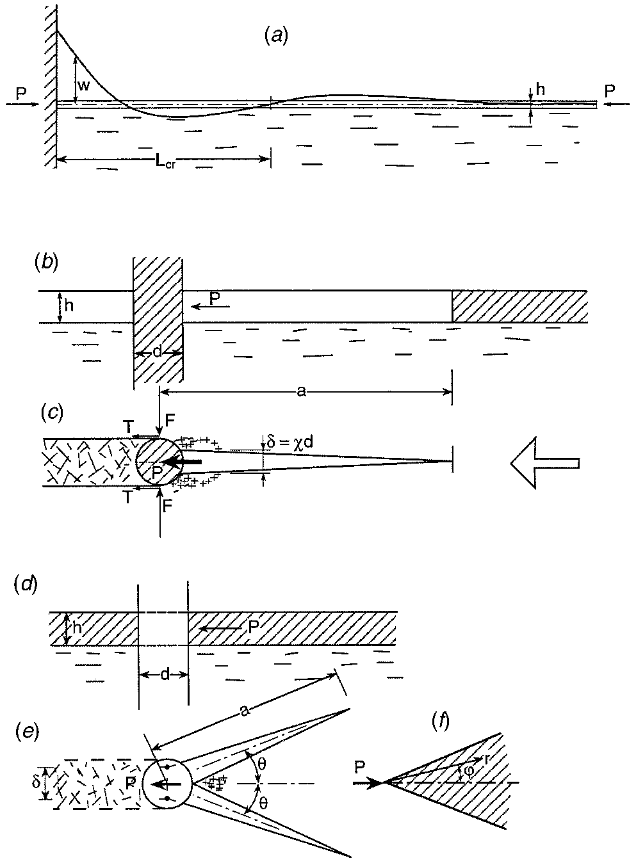

Fig. 1 (*a*) Buckling of ice plate pushing horizontally against a fixed structure, (*b,c,d*) radial cleavage crack, and (*e–g*) diverging *V*-cracks

3 Global Failure due to Radial Cleavage Fracture

Another failure mechanism consists of a long radial cleavage crack in the ice plate, propagating against the direction of ice movement (Fig. 1(*b,c*)). The resistance of the plate against being cleaved causes the ice to exert on the structure a pair of transverse force resultants F and a pair of tangential forces T in the direction of movement; $T = F\tan\varphi$ where φ may be regarded as the friction angle. Forces T have no effect on the stress intensity factor K_I at the crack tip.

First we will consider the asymptotic case of a a structure of a very large size d producing a crack of a very long length a (Fig. 1(*b,c*)). LEFM must apply in this asymptotic case. To determine a, we need to calculate the crack opening δ caused by F. Considering the ice plate as infinite, we have ([12,13])

$$K_I = \frac{F}{h}\sqrt{\frac{2}{\pi a}}. \tag{5}$$

The energy release rate is

$$\mathcal{G} = \frac{1}{h}\left[\frac{\partial \Pi^*}{\partial a}\right]_F = \frac{1}{h}\frac{\partial}{\partial a}\left[\frac{1}{2}C(a)F^2\right] = \frac{F^2}{2h}\frac{dC(a)}{da} \tag{6}$$

where $C(a)$ is the load-point compliance of forces F. Upon using (5) and Irwin's relation ([14]), we have at the same time

$$\mathcal{G} = \frac{K_I^2}{E} = \frac{2F^2}{\pi E h^2 a}. \tag{7}$$

Equating (6) and (7), we thus get

$$\frac{dC(a)}{da} = \frac{4}{\pi E h a}. \tag{8}$$

This expression may now be integrated from $a = d/2$ (the surface of the structure, considered as circular, Fig. 1(*b,c*)) to a (note that integration from $a = 0$ would give infinite C but would be meaningless because a cannot be less than d). In this manner, we obtain $C(a)$, and from it the opening deflection δ:

$$\delta = C(a)F = \frac{4F}{\pi E h}\ln\left(\frac{2a}{d}\right). \tag{9}$$

If the radial cleavage fracture were the only mode of ice breaking, we would have $\delta = d$. However, as will be discussed later, there is likely to be at least some amount of local crushing at, and ahead, of the structure. Consequently, the relative displacement between the two flanks of the crack is no doubt less than d. We denote it as χd where χ is a coefficient less than 1. Upon setting $\delta = \chi d$, (9) yields

$$a = \frac{d}{2}\exp\left(\frac{\pi E h \chi d}{4F}\right). \tag{10}$$

(note that a/d is not constant but increases with d; hence, the fracture modes are not geometrically similar, and so the LEFM power scaling cannot be expected to apply). Substituting (10) into (5) and setting $K_I = K_c = \sqrt{EG_f}$ (Irwin's relation, K_c = fracture toughness of ice), we obtain

$$\frac{2F}{h\sqrt{\pi E G_f d}} = \exp\left(\frac{\pi E h \chi d}{8F}\right). \tag{11}$$

The pair of forces F is related to load P on the structure ($P = 2T$, Fig. 1(*c*)) by a friction law, which may be written as

$$P = 2F\tan\varphi. \tag{12}$$

where φ is the friction angle. Substituting $F = P/2\tan\varphi$ and $P = \sigma_N h d$ into (11), we obtain, after rearrangements,

$$\frac{d}{dc} = \frac{1}{\tau^2}e^{1/\tau}, \quad \tau = \frac{\sigma_N}{\eta_c} \tag{13}$$

in which τ is the dimensionless nominal strength, and d_c and σ_c are constants defined as

$$d_c = \frac{4G_f}{\pi\chi^2 E}, \quad \sigma_c = \frac{\pi}{2}\chi E\tan\varphi. \tag{14}$$

Equation (13), plotted in Fig. 2, represents the law of radial cleavage size effect in an inverted form. The small-size asymptotic behavior is the LEFM scaling for similar structures with similar cracks:

$$\text{for } d \ll d_c: \quad \sigma_N \approx \sqrt{d_c/d}. \tag{15}$$

The plot of (13) in Fig. 2 shows that the size effect is getting progressively weaker with increasing structure diameter d (although no horizontal asymptote is approached by the curve). The reason for this is that the crack is dissimilar, i.e., the ratio, a/d, of crack length to structure diameter is not the same for different sizes but increases according to (10) with the structure size. (In designing ocean platforms, it is nevertheless always advantageous, with respect to the radial cleavage mechanism, to use a smaller number of larger legs, which has of course been intuitively followed in practice.)

So far our radial cleavage crack analysis has been based on LEFM. In other words, the length $2c_f$ of the cohesive zone at the tip of the radial cleavage crack was considered negligible compared to a. Let us now consider the opposite asymptotic case of a very small structure diameter d and a very short crack a such that $a \ll c_f$. In that asymptotic case, the crack faces up to $x = a_0 = a - d/2$ are subjected to uniform cohesive tractions f_t'. Noting that the stress intensity factor for a semi-infinite crack in an infinite

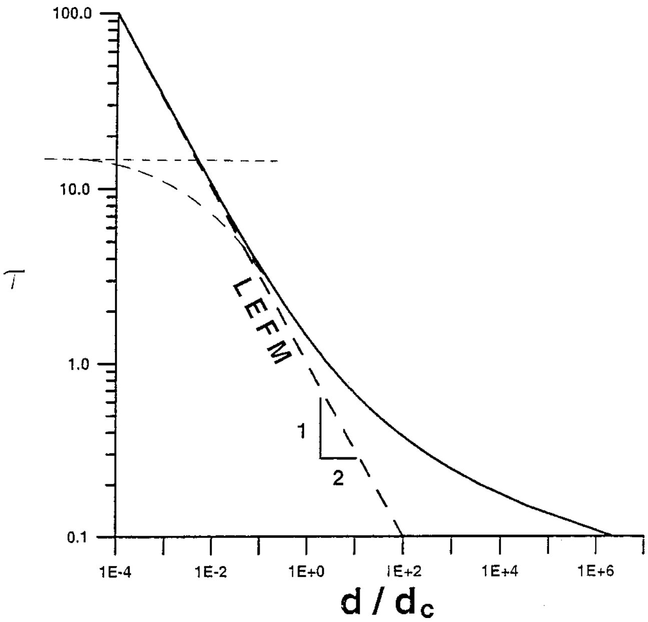

Fig. 2 Size effect associated with radial cleavage fracture (solid curve—LEFM solution, dashed curve—cohesive crack solution)

space caused by a pair of unit concentrated force acting on the crack faces at distance x from the crack tip is $K_I = (\sqrt{2\pi/x})/h$ ([12]), we find that K_I caused by uniform tractions f_t' is

$$K_{I_f} = -\int_0^{a-d/2} \sqrt{\frac{2}{\pi x}}\, f_t' dx = -f_t' \sqrt{\frac{8a}{\pi}}. \tag{16}$$

The stress intensity factor due to concentrated reaction F at distance a from the cohesive crack tip is, according to (5), $K_{IF} = (F/h)\sqrt{2/\pi a}$. It is necessary that the total stress intensity factor $K_{It} = K_{IF} + K_{I_f} = 0$. From this condition and the friction relation (12), it follows that

$$P = 4ahf_t' \tan\varphi. \tag{17}$$

To calculate the deflection δ_f due to cohesive stresses f_t', one could use Green's function. However, this leads to a complicated integral. Since a high accuracy is not needed, we prefer an approximate calculation. To this end, we imagine the cohesive crack length x to grow from 0 to $a_0 = a - d/2$ while constant tractions f_t' act along the entire crack length in front of the structure (and work on the growing opening). In view of (16) and Irwin's relation, the total energy released during the imagined growth of this crack is

$$\Pi^* = \int_0^{a_0} \frac{K_I^2}{E} dx = \int_0^{a_0} \frac{1}{E}\left(f_t' \sqrt{\frac{8x}{\pi}}\right)^2 dx = \frac{4f_t'^2 a_0^2}{\pi E} \tag{18}$$

which is a function of f_t' representing the complementary energy. According to Castigliano's theorem, differentiation of Π^* with respect to the total cohesive force $a_0 h f_t'$ provides the displacement parameter on which the cohesive stress f_t' works, which is the average crack-opening displacement $\overline{v_f}$ over the length a_0 of application of f_t';

$$\overline{v_f} = \frac{1}{a_0 h} \frac{\partial \Pi^*}{\partial f_t'} = \frac{8f_t' a_0}{\pi E h}. \tag{19}$$

Since we avoided Green's function, we now need to approximate the relationship between $\overline{v_f}$ and opening displacement v_f at the center of the structure, $x = a$ (Fig. 1(c)). We may assume that the face of the opened crack is approximately straight, which is a simplification widely used in materials science. Under that assumption, $v_f = \kappa_f \overline{v_f} a/a_0$, where $ka_f = 2$ if the crack face remains straight. Then, from (19),

$$v_f = \frac{8f_t'}{\pi E} \kappa_f a. \tag{20}$$

The opening displacement $2v_F$ due to the pair of concentrated forces F has already been calculated in (9); $2v_F = (4F/\pi Eh)\ln(2a/d)$. Compatibility of transverse displacements at the center of structure $(x = a)$ requires that

$$2v_F - 2v_f = \chi. \tag{21}$$

Substituting the foregoing expressions for v_F and v_f, and setting $F = P/2\tan\varphi = \sigma_N hd/2\tan\varphi$, one obtains, after rearrangements, the equation

$$\ln\left(\frac{\sigma_N}{2\, f_t' \tan\varphi}\right) = \kappa_f + \frac{\pi E \chi \tan\varphi}{2\sigma_N}. \tag{22}$$

This is a transcendental equation whose solution $\sigma_N = \sigma_N^0$, represents the average pressure applied on the area of the structure facing the moving ice. Since d and h do not appear in this equation, the σ_N^0 value is a constant, represented in Fig. 2 by the horizontal line. So, as expected, there is asymptotically no size effect if $d \to \infty$.

To obtain the approximated law of the size effect for the intermediate sizes, the small-size and large-size asymptotic behaviors must be suitably matched. Similar to many previous approximations of quasi-brittle size effect ([13,14]), the asymptotic matching may be accomplished by replacing size d in (13) with the expression $(d^r + d_0^r)^{1/r}$ where d_0 is a constant. With this replacement, (13) provides the following general approximate asymptotic matching law for the size effect:

$$(d^r + d_0^r)^{1/r} = \frac{d_c}{\tau^2} e^{1/\tau}. \tag{23}$$

Here r is an empirical constant, probably close to 1. For $d \to \infty$, this equation asymptotically approaches the LEFM Eq. (13), and for $d \to 0$ the following equation for constant d_0 is obtained:

$$d_0 = \alpha_c (\sigma_c^2 / \sigma_N^{0\,2}) e^{\sigma_c/\sigma_N^0} \tag{24}$$

where σ_N^0 is the solution σ_N of (22). In analogy to other scaling problems, the value $r = 1$ is often reasonable, and then (23) simplifies to the size effect formula:

$$d = \frac{d_c}{\tau^2} e^{1/\tau} - d_0. \tag{25}$$

Equation (23) for the quasi-brittle size effect due to a horizontal load is plotted as the dashed curve in Fig. 2. The shape of this plot documents the difficulty in deducing the size effect from small-scale experiments. If the tests are confined to the nearly horizontal intial portion of the dashed curve, there is no way to predict the size effect at large sizes unless a realistic theory is employed.

4 Compression Fracture of Ice Plate

As typically observed in the field, moving ice gets crushed in front of an obstacle, breaking up into chunks. The cause is local compression fracture of the material. Its initiation may be explained by sliding on inclined weak plains between ice crystals, which leads to axial splitting microcracks called the wing-tip cracks (for ice see, e.g., Schulson [15,16]) extending in the direction of compression for a certain finite length. This mechanism, however, explains only the generation of local compressive damage in the material but does not explain to overall failure of the plate and the size effect.

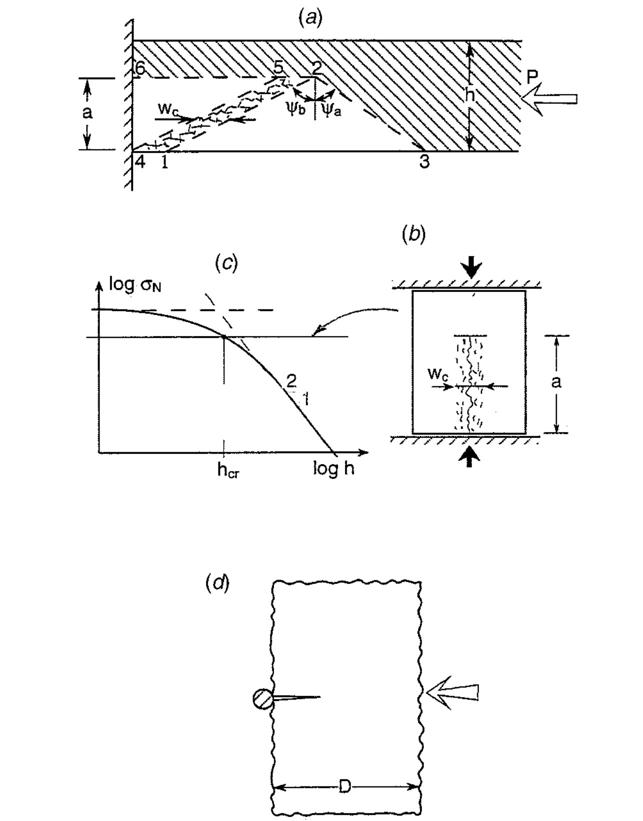

Fig. 3 (*a*) Compression fracture of ice plate, (*b*) axial splitting fracture, (*c*) size effects corresponding to (*a*) and (*b*), and (*d*) overall fracture of ice floe

To produce overall breakup of ice, the damage must propagate. As transpired in connection with studies of concrete and borehole breakout in rock ([13,17]), the propagation typically occurs in the form of a narrow band consisting predominantly of axial splitting microcracks (generated, e.g., by the wing-tip crack mechanism). The band of axial splitting microcracks can propagate either in the axial direction of the compressive stress, or laterally. The latter is shown in Fig. 3(*a*), and the former in Fig. 3(*b*).

In the spirit of fracture mechanics, one must estimate the energy release. Consider the plausible situation depicted in Fig. 3(*a*), where the band of a certain characteristic width w_c in the direction of compression has inclination ψ_b and reaches to depth a below the surface of plate. Formation of the band must evidently relieve the axial stress σ_N not only within the band area 12541, but also in the adjacent zones 1231 and 4564. The boundary of the stress relief zone is considered to have a certain characteristic inclination ψ_a, independent of the plate thickness. The combined area of the stress relief zone 43264 is $a(w_c + a/2 \tan\psi_a + a\tan\psi_b)$. Before the formation of the damage band, the initial strain energy density in this zone is $\sigma_N^2/2E$, and after the formation of the band it may be assumed as zero (more generally, one could quite easily take into account some finite residual strength σ_r of ice after crushing, see [17]; but this is omitted since no information on σ_r is available). Thus the total energy release caused by formation of the damage band per unit width is, approximately,

$$\Pi^* = \frac{\sigma_N^2}{2E} a\left(w_c + \frac{1}{2} a \tan\psi_a + a\tan\psi_b\right). \qquad (26)$$

The rate of energy dissipation per unit width as the band propagates must be equal to the fracture energy of the band, G_b, which equals $G_f w_c/s_c$ where G_f is the fracture energy of the axial splitting microcracks in the band, and s_c their average spacing. Energy balance during the quasi-static extension of the band requires that the rate of energy release be equal to G_f, i.e.,

$$\frac{\partial \Pi^*}{\partial a} = \frac{\sigma_N^2}{2E}(w_c + a\tan\psi_a + 2a\tan\psi_b) = G_b. \qquad (27)$$

Solving this equation for σ_N, we get, after rearrangements,

$$\sigma_N = \sigma_a\left(1 + \frac{h}{h_0}\right)^{-1/2} \qquad (28)$$

in which the following notations are made:

$$h_0 = \frac{h}{a}\frac{w_c}{(\tan\psi_a + 2\tan\psi_b)}, \quad \sigma_a = \sqrt{\frac{2EG_c}{w_c}}. \qquad (29)$$

Here we deliberately introduced the plate thickness h even though it cancels out of the equation. The reason is that it appears reasonable to assume the ratio a/h for plates of various thicknesses to be approximately constant. In other words, the geometries of the damage band at failure of the plates of various thicknesses are assumed similar. This assumption is based on experience with some other fracture problems, for which it was shown to lead to realistic results. Anyway, it is intuitively clear that it would be unreasonable to assume that for thin plates the damage band at maximum σ_N penetrates through most of the thickness and for thick plates penetrates only to a very shallow depth.

Equation (28), plotted in Fig. 3(*c*), is the same as the classical size effect law proposed by Bažant [18] for quasi-brittle structures failing after a long stable growth of tensile fracture. Among the mechanisms explored here, it is the only one that can explain the size effect of ice thickness.

The ultimate cause of size effect in compressive (as well as tensile) fracture is that the volume of the energy dissipation zone, i.e., the damage band, grows linearly with the distance a of propagation while the volume of the energy release zone grows faster than linearly, having a quadratically growing term that dominates for large sizes. Thus it is intuitively clear that if the stress in these zone at failure were the same, energy balance could exist only for one size but not for other sizes ([14]). So, in a larger structure the stress in the quadratically growing zones (1231 and 4564 in Fig. 2(*a*)) must be less.

There is of course another possibility—namely that the damage band grows axially, in the direction of compression, which leads to a splitting failure (Fig. 3(*b*)). In that case the stress in the material on the sides of the crack band is not relieved, and so the energy release occurs only within the damage band itself. In that case, not only the energy dissipation but also the energy release are proportional to the length a of the band, which means that energy rates for the same failure stress σ_N can balance for any size h. So, for the axial propagation, there is no size effect.

The axial growth is more likely because no new wing-tip cracks need to be nucleated. Therefore, at small enough sizes the axial splitting of ice should prevail, which means that the splitting mechanism corresponds in the logarithmic size effect plot (Fig. 3(*c*)) to a horizontal line starting below the curve of the size effect law for lateral propagation of the damage band. However, the horizontal line must eventually cross the size effect curve at a certain critical size h_{cr}, above which the lateral propagation of damage band must prevail, and then a size effect must exist.

The present analysis is similar to that made for concrete; see [17], where various fine details are discussed (also [13,14]).

Finally, an explanation of empirical parameter χ introduced for the cleavage fracture: It is presumed that the part $(1-\chi)d$ of the cross section facing the ice movement undergoes compression crushing. This part should be governed by Eq. (28), and so the force given by that equation needs to be added to the force P based on (12) and (13).

5 Overall Fracture of Finite Ice Floe

Collision of a large ice floe with a fixed structure may cause a fracture of the whole floe. The floe is loaded by distributed inertia forces of its mass, but the problem may be treated as essentially quasi-static, owing to the low velocity of movement. Except for the loading by distributed forces, the problem is similar to fracture tests in the laboratory, especially the three-point bend beam (Fig. 3(*d*)). Dempsey's record-breaking tests on the Arctic Ocean near Resolute can be regarded as an approximate reduced-scale simulation of this kind of fracture ([19,20]). The analysis may follow similar lines as presented, for instance, in [13] for other materials. From that analogy it follows that the size L of the floe may cause one of two types of size effect:

$$(1)\quad \frac{P}{Lh}=S_0\left(1+\frac{L^r}{L_0^r}\right)^{-1/2r} \tag{30}$$

$$(2)\quad \frac{P}{Lh}=S_\infty\left(1+\frac{rL_b}{L}\right)^{1/r} \tag{31}$$

where P/Lh is the nominal strength of the whole floe; S_0, L_0, S_∞, L_b are constants that can be calculated by fracture mechanics; and r is a parameter whose value is normally between 0.5 and 2.

The first kind of size effect, which agrees very well with Dempsey et al.'s [19] field tests in the Arctic, applies when a large crack in the floe can form before the overall fracture of the floe takes place. The second kind applies to failures at fracture initiation, exemplified by the test of modulus of rupture (bending strength), and is pertinent if the maximum load is attained before a stable finite crack can develop (e.g, by means of the radial cleavage mechanism).

6 Comments on Some Periodic Failure Mechanisms

According to observations, diverging V-shaped cracks may also form ahead of an obstacle (e.g., [21], ch. 7); Fig. 1(*e*,*f*). To estimate in a simple manner a rough approximate value of complementary energy Π^* of an infinite ice plate after formation of such cracks, we may assume that the force P from the structure produces stress only within the wedge between the cracks (Fig. 1(*g*)). From a well-known solution ([22]),

$$\sigma_r=-Pk_\theta\cos\varphi/rh,\quad \sigma_\varphi=\sigma_{r\varphi}=0 \tag{32}$$

where σ_r, σ_φ, and $\sigma_{r\varphi}$ are the stress components in polar coordinates r, φ, and

$$k_\theta=1\bigg/\left(\theta+\frac{1}{2}\sin 2\theta\right), \tag{33}$$

θ being the inclination angle of the cracks (Fig. 1(*f*)). The displacement at $r=d/2$ (structure surface) is

$$u=\int_{d/2}^{\infty}\frac{\sigma_r}{E}dr=\frac{Pk_\theta}{Eh}\ln\frac{2a}{d}. \tag{34}$$

Then $\Pi^*=Pu/2=(P^2k_\theta/2Eh)\ln(2a/d)$. The complementary energy before fracture may be estimated as the value of Π^* for $\theta=\pi$, i.e., $\Pi_0^*\approx(P^2/2\pi Eh)\ln(2a/d)$. The total energy release due to V-cracks in the ice plate is $\Delta\Pi^*=\Pi^*-\Pi_0^*$, and the derivative $\partial\Delta\Pi^*/\partial a$ at constant P must be equal to $2hG_f$. This condition yields

$$P\approx 2h\sqrt{\frac{EG_f}{\pi^{-1}-k_\theta}}\sqrt{a}. \tag{35}$$

To determine crack length a and angle θ, one may use two conditions: (a) the opening displacement at the crack mouth, δ, must be equal to $\chi d/(2\cos\theta)$, which means that the load-point displacement of force P must be $u=(\chi d/2)\tan\theta$, and (b) the expression for P should be minimized with respect to θ. These two conditions, however, make the solution quite complicated. We will not pursue it here because of this and also because of two unresolved questions: (1) An axial cleavage crack may be also present ([5]), and it may form either before or after the V-cracks. (2) Simultaneous compression crushing is very likely in the case of V-cracks, which makes the value of χ, and thus the length a of V-cracks, rather uncertain.

Unlike the cleavage fracture, the V-shaped cracks can occur only from time to time. They do not represent a steady-state mechanism that would accommodate continuous movement of the ice.

Other failure mechanisms occur in the case of an inclined face of the fixed structure, or in the case of an icebreaker ([5]). These mechanisms involve axial bending cracks as well as bending cracks normal to the direction of motion. Studying the action of an icebreaker, Goldstein and Osipenko [3] considered periodic formation of LEFM bending cracks at some distance in front of the icebreaker, normal to the direction of movement. They limited attention to one-dimensional cylindrical bending of the ice plate and did not consider simultaneous formation of axial or other cracks.

7 Conclusions (From Parts I and II)

1. The known mechanism of failure of a floating ice plate subjected to a vertical load can be used in an approximate energy analysis of quasibrittle fracture. The results do not disagree with the limited field experiments that exist. They approximately agree with previous numerical simulations and confirm that for large ice thicknesses there is a strong size effect, approaching the size effect of LEFM. Asymptotic matching leads to a simple formula for the size effect, which is similar to the size effect law proposed in 1984 by Bažant.
2. Simplified fracture analysis of the nominal strength of ice plate pushed against a fixed structure brings to light several possible mechanisms of failure with size effects due to ice thickness, the diameter of the structure and, if the size of the ice floe is finite, the size of the floe. Buckling of the floating plate causes a reverse size effect of ice thickness (i.e., the nominal strength increasing with ice thickness) and therefore plays any role only for sufficiently thin ice. Radial cleavage of the ice plate against the direction of ice movement causes a size effect of structure diameter which follows linear elastic fracture mechanics (LEFM) for small enough diameters and becomes progressively weaker with an increasing diameter. Compression fracture, with ice crushing localized into transversely propagating bands, causes a size effect of ice thickness that follows approximately the classical size effect law proposed in 1984 by Bažant. The overall fracture of a finite ice floe causes a size effect of the floe size, following again the same size effect law.
3. The present approach contrasts with the classical approach based on either plastic limit analysis or elastic analysis with a strength limit, both of which lead to no size effect.

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