Fracturing Rate Effect and Creep in Microplane Model for Dynamics

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Abstract: The formulation of microplane model M4 in Parts I and II is extended to rate dependence. Two types of rate effect in the nonlinear triaxial behavior of concrete are distinguished: (1) Rate dependence of fracturing (microcrack growth) associated with the activation energy of bond ruptures, and (2) creep (or viscoelasticity). Short-time linear creep (viscoelasticity) is approximated by a nonaging Maxwell spring-dashpot model calibrated so that its response at constant stress would be tangent to the compliance function of model B3 for a time delay characteristic of the problem at hand. An effective explicit algorithm for step-by-step finite-element analysis is formulated. The main reason that the rate dependence of fracturing must be taken into account is to simulate the sudden reversal of postpeak strain softening into hardening revealed by recent tests. The main reason that short-time creep (viscoelasticity) must be taken into account is to simulate the rate dependence of the initial and unloading stiffness. Good approximations of the rate effects observed in material testing are achieved. The model is suitable for finite-element analysis of impact, blast, earthquake, and short-time loads up to several hours duration.

INTRODUCTION

Changes of the loading rate affect stress-strain relations for concrete in two ways: (1) Through the rate dependence of the growth of the distributed microcracks in concrete, and (2) through viscoelasticity or creep of the material between the cracks. For some materials (e.g., ceramics), the former mechanism dominates, while for others (e.g., some polymers) the latter dominates. Analysis of certain recent experiments (Bažant and Gettu 1992; Bažant 1993, 1995; Bažant et al. 1993, 1995; Bažant and Jirásek 1993; Tandon et al. 1995; Bažant and Li 1997) showed that for concrete both mechanisms are important, except that the former dominates at the extreme strain rates under impact. Both mechanisms have been extensively analyzed for concrete.

The literature contains many important studies of the rate effects in various materials [e.g., Evans (1942); Watstein (1953); Hughes and Gregory (1972); Sparks and Menzies (1973); Mindess (1985); Reinhardt (1985); Liu et al. (1989); Kanstadt (1990); Ross and Kuennen (1989); Sluys (1992); You et al. (1992); Zhou and Hillerborg (1992); see also the reviews in Bažant and Oh (1982), and Bažant and Planas (1998). Although this study is focused on stress-strain relations, which describe a material with many microcracks but no distinct large cracks, the findings on time dependence of the growth of a macroscopic crack are relevant too, since the growth of microcracks must obey the same laws. A large body of knowledge has been accumulated on the time dependence of fracture of polymers (Williams 1963, 1964, 1965; Schapery 1975, 1978, 1982, 1984, 1988, 1989; Knauss 1970, 1989, 1993a,b; Bažant and Li 1997) as well as ceramics [e.g., Evans 1942; Thouless et al. 1983; Evans and Fu 1984].

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In some earlier studies (e.g., Mihashi and Izumi 1977, Zech and Wittmann 1977; Wittmann 1985), the rate effects in concrete were explained by Weibull type statistical effects of strength randomness. The question of possible statistical effects, however, is left out of the present study. In a certain sense, statistical effects are implied in the activation energy theory (Krausz and Krausz 1988), and other statistical effects seem to be minor compared to the rate dependence of crack opening and creep.

In simulations of missile impact and penetration with a time-independent constitutive law, the microplane model for concrete has proven to be significantly more successful than the classical plasticity models. Model M4, which is the last version of the microplane models developed at Northwestern University (Bažant et al. 2000a; Caner and Bažant 2000), as well as the preceding model M3 (Bažant et al. 1996a,b), has been found at the U.S. Army Waterways Experiment Station (WES) to be the only constitutive model for concrete that is able to correctly simulate the deformation and fracturing patterns observed in penetration experiments, including the smaller entering crater and the larger exit crater. Despite determined efforts at WES, this has not been achieved with the classical constitutive models expressed in terms of tensorial invariants and based on plasticity or phenomenological continuum damage mechanics. The microplane model has also provided the best results for the blast and groundshock effects on reinforced concrete structures, predicting correct final deflections and cracking patterns. Without the rate effects, however, model M4 calibrated by extensive static triaxial test data predicted excessive exit velocities of a missile penetrating a wall. The model parameters could of course be easily adjusted to fit the exit velocities, in disregard of static material tests, but such a model would not have predictive power.

The main reason for the errors in exit velocities in these simulations has doubtless been the lack of rate dependence of fracturing. The fracturing is simulated by microplane models as smeared microcracking, with large fractures modeled as crack bands. The growth of microcracks is described by fracture mechanics. Rate independent though the theory of fracture mechanics is, crack propagation nevertheless is always rate dependent. Cracks cannot form truly instantly, but open and propagate with a certain finite speed.

The rate dependence of the opening of cohesive cracks, or the rate dependence of crack length growth, has recently been shown to be the cause of the reversal of softening into hardening after a sudden increase of the loading rate [tests by Bažant et al. (1995), Tandon et al. (1995); modeled by Bažant and Jirásek (1993); Bažant and Li (1997)]. This means that the tangent modulus for loading can suddenly change from negative to positive as the loading rate is suddenly increased (or, more generally, the effective tangential stiffness matrix of the microcracked material can suddenly change from nonpositive definite to positive definite).

One implication of this recently discovered new phenomenon is that a material that is in strain-softening state can propagate not only unloading waves but also loading waves with a sufficiently steep front, which is typical of impact (loading waves with a sufficiently mild front are of course unable to propagate, except for a limited spreading due to the necessity of nonlocal behavior of strain softening). This phenomenon is probably important for simulating crater formation farther away from the missile because it enables the loading stress waves to be transmitted through the concrete near the missile in which strain softening has already begun. As for unloading waves, they can of course always propagate through a material that is in a strain softening state because the loading modulus is always positive. So the modeling of the softening-hardening reversal appears important to render the simulation of complete cratering possible.

RATE DEPENDENCE OF CRACK OPENING

In similarity to the formulation proposed in Bažant (1993), justified in more detail in Bažant (1995), and used in Bažant and Li (1997) and Li and Bažant 1997), the rate dependence of the opening w of a cohesive crack (also called ''fictitious'' crack) may be described as

$$\dot{w} = k_0 \sinh\left(\frac{T_0}{T}\frac{\sigma - f^0(w)}{k_r N_b(w)}\right) e^{-(Q/R)(1/T - 1/T_0)} \tag{1}$$

[Fig. 1(a)]; $\dot{w} = dw/dt$, t = time; σ = cohesive (crack-bridging) stress; k_0, k_r, T^0, Q, and R = constants (R = universal gas constant; Q = activation energy of interatomic bond ruptures; T = absolute temperature; T_0 = given reference temperature; and $N_b(w)$ = number of surviving bonds across the cohesive crack per unit area). Function $f^0(w)$ describes the softening law of the cohesive crack at an opening rate that is at the lower limit of the range of rates for which (1) is to be applied or has been calibrated (normally the rate of loading in static material tests in the laboratory). The adiabatic heating of concrete in an impact event probably has a negligible effect on $\dot{w}$, and in that case one may set $T \approx T_0$.

Eq. (1) can be imagined to correspond to the rheologic model in Fig. 1(b), in which a rate-independent cohesive crack element is coupled in parallel with a nonlinear damper at each point of the cohesive crack. The sinh-function in (1) ensues from the activation energy theory or rate-process theory (Krausz and Krausz 1988) for bond ruptures, as shown for concrete in Bažant (1993) and in more detail in Bažant (1995). This function makes the rate effect highly nonlinear, which is important when many orders of magnitude of the loading rate are to be modeled.

Various linear stress-displacement relations, analogous to spring-dashpot models of viscoelasticity, have been proposed by Kanstadt (1990); Zhou and Hillerborg (1992); Sluys (1992); and others. But they perform realistically for only one order of magnitude of the loading rate, while (1) is applicable over many orders of magnitude.

The ratio $\sigma/N_b(w)$ represents the transmitted stress per bond, which is what matters for the activation energy theory. Obviously, N_b must decrease with increasing crack opening w and must drop to 0 when w becomes so large that $\sigma = f(w) = 0$. As an approximation, it seems reasonable to assume that $N_b(w) = C_b f^0(w)$ where C_b is some proportionality constant.

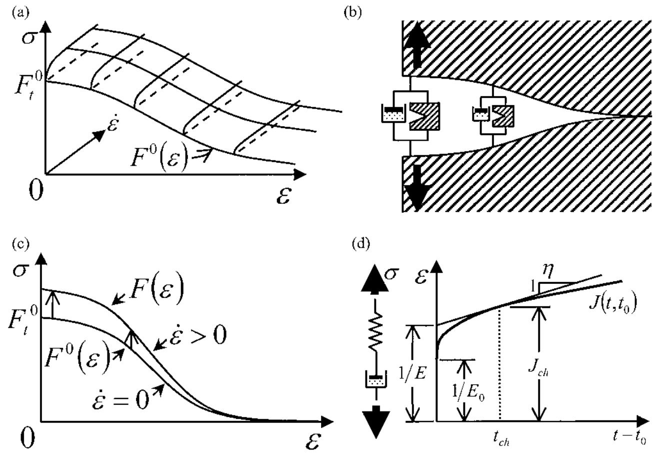

FIG. 1. (a) Softening Curves of Crack-Bridging Stress versus Smeared Cracking Strain at Different Rates of Loading According to (1); (b) Rheologic Model Illustrating Eq. (1); (c) Vertical Scaling of Stress-Strain Boundary; (d) Approximation of Actual Compliance Function by Tangent Line Corresponding to Effective Maxwell Model

RATE DEPENDENCE OF MICROPLANE STRESS-STRAIN BOUNDARIES

The macroscopic strain softening may be imagined to be the result of the openings on many parallel cohesive cracks. Denoting by s_{cr} their average spacing, one may introduce the approximation $\dot{\varepsilon} = \dot{w}/s_{cr} + \dot{\sigma}/E \approx \dot{w}/s_{cr}$ where ε = average macroscopic strain normal in the direction normal to the parallel cracks; E = Young's modulus of uncracked material; and $\dot{\sigma}/E$ represents the elastic strain rate, which is normally negligible by comparison if the crack is opening.

Substituting $\dot{w} = s_{cr}\dot{\varepsilon}$ and $N_b(w) = C_b f^0(w)$ into (1) and solving the equation for σ, one obtains $\sigma = F(\varepsilon)$, where

$$F(\varepsilon) = F^0(\varepsilon)[1 + C_2 \operatorname{asinh}(\dot{\varepsilon}/C_1)] \tag{2}$$

$$\text{with} \quad C_1 = C_0 e^{-(Q/R)(1/T - 1/T_0)}; \quad C_2 = C_r T/T_0; \quad \dot{\varepsilon} \ge 0 \tag{3a--c}$$

and the notations $C_0 = k_0/s_{cr}$, $C_r = k_r C_b$ and $f^0(w) = F^0(\varepsilon)$; C_0 and C_r are constants to be determined from tests. When the temperature is about the same as the temperature in the laboratory tests used for calibration of the present model, which is about 25°C (T_0 = 298°K), then $C_1 = C_0$ and $C_2 = C_r$, and C_1 and C_2 become temperature independent, that is, constants.

Since $\operatorname{asinh} x = \ln(x + \sqrt{x^2 + 1})$, the asymptotic approximation for $x^2 >> 1$ is $\operatorname{asinh} x \approx \ln(2x)$. In (2), this occurs if the loading rate is so large that $\dot{\varepsilon}^2 >> C_1$. Then (2) takes the form:

$$F(\varepsilon) \approx F^0(\varepsilon)[1 + C_2 \ln(2\dot{\varepsilon}/C_1)] \tag{4}$$

The logarithmic function is normally used for the rate effect on the yield limit in metal plasticity (discussed later in the section on numerical implementation in the wave code ''EPIC'').

Function $F(\varepsilon)$ may be interpreted as the stress-strain boundary on the microplane corresponding to strain rate $\dot{\varepsilon}$, and $F^0(\varepsilon)$ has the meaning of the static stress-strain boundary that corresponds to a vanishing strain rate (a boundary that approximately applies to the small loading rates of static material tests). The transformation from the static boundary $F^0(\varepsilon)$ to the rate-dependent (dynamic) boundary represents, according to (2), a vertical scaling of the boundary curve [Fig. 1(c)].

In microplane model M4, there are various types of stress-strain boundaries controlling the inelastic behavior. On each microplane, there are boundaries for tensile normal stress σ_N; compressive and tensile volumetric stresses σ_V; tensile or compressive deviatoric stresses σ_D; and shear stress σ_T [Bažant et al. 2000a, Eq. (5); Bažant et al. 1996a, Eqs. (12), (20)–(22)]. They are denoted as F_N, F_V, F_V^+, F_D, F_D^+, and F_T [and are used in the microplane stress-strain relations in Eqs. (14)–(20) of Bažant et al. 2000a, or Eqs. (13)–(19), Bažant et al. 1996b]. Their arguments are the microplane strain components ε_N, ε_V, ε_D, and ε_T, which represent the microplane projections of the macroscopic strain tensor $\boldsymbol{\varepsilon}$.

It is logical to assume that the normal and deviatoric boundaries adhere to (2). Thus, with superscript 0 labeling the boundaries for static loading (vanishing strain rate), one may write

$$F_N(\varepsilon_N) = F_N^0(\varepsilon_N)[1 + c_{R2}R(\dot{\gamma})] \tag{5}$$

$$F_D(\varepsilon_D) = F_D^0(\varepsilon_D)[1 + c_{R2}R(\dot{\gamma})] \tag{6}$$

$$F_D^+(\varepsilon_D) = F_D^{0+}(\varepsilon_D)[1 + c_{R2}R(\dot{\gamma})] \tag{7}$$

where, however, $\operatorname{asinh}(\dot{\varepsilon}/C_1)$ is replaced by function $R(\dot{\gamma})$ of global strain rate measure $\dot{\gamma}$, in view of certain tensorial invariance aspects. The following simple definition is proposed:

$$R(\dot{\gamma}) = \operatorname{asinh}(\dot{\gamma}/c_{R1}) \approx \ln(2\dot{\gamma}/c_{R1}) \quad \text{with} \quad \dot{\gamma} = \sqrt{\frac{1}{2}\,\dot{\varepsilon}_{ij}\dot{\varepsilon}_{ij}} \tag{8}$$

where the logarithmic approximation of R is admissible only if $\dot{\gamma}^2 >> c_{R1}$; c_{R1}, c_{R2} are material rate constants, analogous to C_1, which have been calibrated by test data (they can be considered as fixed parameters, which need not be adjusted by the user and are applicable to all concretes); $\dot{\varepsilon}_{ij}$ are strain rate components with subscripts i, j (=1, 2, 3) referring to Cartesian components, and repetition of subscripts implies summation; and $\dot{\gamma}$ is a nonnegative invariant of $\dot{\varepsilon}_{ij}$ (note that the square root of the second invariant of the strain rate tensor would be inappropriate because the cracking strain involves volume change).

Eq. (8) means that the rate magnitude is measured not individually for each microplane and each component, but globally for all the microplanes, that is, on the macroscale. The reason is the objectivity of modeling. A microplane strain rate component such as $\dot{\varepsilon}_N$ or $\dot{\varepsilon}_D$ does not simply characterize the average opening rate of all the microcracks parallel to that microplane. Rather, it represents merely the contribution from that microplane to the overall cracking rate, whose dominant orientation is in general different. So $\dot{\varepsilon}_N$ and $\dot{\varepsilon}_D$ consist in effect of the components of this overall cracking strain, but the rate effect may not be based on the rate of a component. Rather, it must be based on the rate of overall cracking. For instance, when the rate of the dominant cracking strain is very high, the contribution to it from a microplane of a certain inclined orientation could be very small, but if the rate effect were decided by the rate of this contribution alone, then an almost static stress-strain boundary would apply on this microplane (because of very low $\dot{\varepsilon}$). Thus, different subdivisions into microplane components would yield different results, which would be unobjective. Also note that separate averaging of, for example, the shear components on all the microplanes would be questionable because $\dot{\varepsilon}_N$ on one microplane is tied by the kinematic constraint to $\dot{\varepsilon}_T$ on a microplane inclined to it.

Computational experience showed that if the averaging of rate effect over all the microplanes is omitted, convergence is impaired and may be lost. Admittedly, such behavior might be taken into account by (8) very crudely, but the existing limited data do not permit calibrating some more sophisticated formulation.

No rate effect obviously exists for the tensile crack-closing boundary. None is introduced for pore collapse described by the compressive volumetric boundary, which seems to be an acceptable simplification. The creep effects are of course included. (Note that, even in saturated concrete, there is no analogy with the "squeezing" of water out of saturated sand because much of the pore water is in the adsorbed state.)

The rate effect on the (σ_T, σ_N) frictional yield surface is not applied to both σ_T and σ_N. Rather, it is introduced as scaling of the yield surface in the vertical (i.e., σ_N) direction, which is simply implemented by a vertical shift of the horizontal asymptote. Eq. (17) of the preceding paper (Bažant et al. 2000) is, therefore, adapted as follows:

$$F_T(-\sigma_N, \dot{e}) = c_9\langle\sigma_N^0 - \sigma_N\rangle\left(1 + \frac{c_9\langle\sigma_N^0 - \sigma_N\rangle}{E_T k_1 c_8[1 + c_{R0}R(\dot{e})]}\right)^{-1} \tag{9a}$$

$$\dot{e} = \sqrt{\frac{1}{2}\,\dot{e}_{ij}\dot{e}_{ij}} \tag{9b}$$

where constants E_T, k_1, c_8, and c_9 are defined in that paper; c_{R0} is a rate constant to be calibrated by test data; $\dot{e}_{ij} = \dot{\varepsilon}_{ij} - \dot{\varepsilon}_{kk}\delta_{ij}/3$ = deviator of strain rate tensor; and $\sigma_N^0 = E_T k_1 c_{11}/(1 + c_{12}\langle\varepsilon_V\rangle)$. In contrast to γ in (8), the rate tensor invariant $\dot{e}$ is defined here in terms of the rate deviator tensor because shear slip is unaffected by the volume change rate.

Since friction is commonly considered as a rate-independent phenomenon, inclusion of the rate effect for the friction boundary might seem uncanny. But studies of rocks (e.g., Dieterich 1978) have shown that the rate effect in frictional slip is strong. They also showed that it is very intricate (Rice and Ruina 1983; Ruina 1983; Gu et al. 1984; Rice and Ben-Zion 1996; Ranjith and Rice 1998), such that a strong simplification is required for the present purposes.

In (9), the same function R as for the normal boundaries is assumed to apply. This is only partly for the sake of simplicity. In principle, since the rate effect on the shear boundary is of a different physical nature, arising from frictional slip rather than tensile cracking, the rate effect should follow a different law. However, if the frictional slip is softening (the shear stress drops with increasing slip), then the slip should localize into a set of shear cracks. The growth of these shear cracks requires bond breakages and thus ought to be controlled by activation energy, too, similar to tensile crack growth. For this reason, the same function R as before is used for the rate effect in microplane shear.

Another noteworthy feature of the frictional yield surface is that, in (9), only σ_T is scaled according to the strain rate, whereas σ_N is not. If σ_N were too, it would not be vertical scaling but radial scaling in the stress space (σ_T, σ_N). Such a scaling seems unrealistic because an increase of the strain rate would increase the value of tensile stress at which an infinitesimal shear stress suffices to initiate slip. Anyway, the existing limited test data do not permit deciding this question (but it should be pondered in the future). Besides, the difference between the vertical and radical scalings becomes negligible once the cohesion is reduced near zero.

Although computations show the frictional rate effect not to affect the uniaxial (unconfined) compression by more than a few percent, this rate effect is dominant for very high confinement, that is, when all the principal stresses are compressive and of high magnitudes. Recent experiments (Bažant et al. 1999) demonstrate that, under very high hydrostatic pressures, concrete can undergo, without any visible fracturing, extreme plastic strains of the order of 100%, with shear angles up to 70°. In problems such as deep missile penetration into a concrete wall, these enormous plastic strains dominate the response. Calculations show that if the rate effect associated with yielding were neglected, the energy dissipation of a penetrating missile would be underestimated. On the other hand, since test data on the frictional rate effect are lacking at present, one must at present resort to calibrating c_{R0} by matching the measured exit velocities with finite-element simulations.

For uniaxial (unconfined) tension, the normal boundary controls the tensile strain softening due to cracking. The postpeak softening in uniaxial (unconfined) compression is controlled mainly by the tensile deviatoric boundary, although the tensile normal boundary and the friction yield surface are also important. Thus, to simulate the strength increase with the rate of loading, the normal and deviatoric boundaries need to be scaled with a function of the normal strain rate characterizing the growth of tensile microcracks. Therefore, the rate effect given by (2) is applicable to both the normal and deviatoric boundaries.

On the other hand, the rate effect on the compressive volumetric boundary is neglected because this boundary simulates neither crack formation nor crack propagation. Since the volumetric tensile boundary comes into play only at very large volume expansion, its rate dependence is also neglected, for the sake of simplicity.

More accurately, the stress-strain boundaries F could be considered as functions of ε_{cr} and $\dot{\varepsilon}_{cr}$, rather than the total strains ε and $\dot{\varepsilon}$, which result by adding the elastic strain or strain rate. However, the total strains are more convenient since they allow explicit expressions for stress as a function of strain (a great advantage for finite-element codes). The error in using the total strains of concrete instead of the cracking strains is negligible because fracture becomes important only after the cracking strain magnitude becomes much larger than the elastic strain magnitude. Besides, the replacement of the inelastic strain rate with the total strain rate is a well-tried simplification in rate-dependent plasticity of metals, widely used in computer codes such as the hydrocode EPIC at WES (discussed later). In those codes, the yield surface might ideally depend on the rate of plastic strain rather than the total strain, but the latter has proven successful.

The horizontal boundaries are scaled vertically in the same ratio as the strain-softening boundaries. This tends to shorten the strain range of the horizontal boundary as the strain rate increases, which produces a sharper peak. However, the increase in the initial effective elastic modulus with increasing strain rate tends to lengthen this range and may prevail. Thus, an increase of loading rate in uniaxial compression may produce either a rounder or a sharper peak, according to the present model.

Since the cracks are closing during unloading, the rate effect of fracturing might be expected to be weaker (although it should not disappear entirely because the crack closing is hindered by debris in the cracks). The fitting of the available test data has nevertheless revealed no need to consider a different rate effect for unloading.

In numerical calculations, the strain rate is approximated as $\dot{\varepsilon} = \Delta\varepsilon/\Delta t$ (where Δt = time step and Δ denotes the increment per step). In an explicit dynamic program such as EPIC, $\Delta\varepsilon$ needs to be taken from the previous time step, which gives only the first-order accuracy in Δt. In an implicit program with iterations of the loading step, $\Delta\varepsilon$ can be taken from the preceding iteration, which raises the accuracy to the second order in Δt. Although the present formulation performs well, future studies should explore various possible alternatives to the strain rate measure in (8); for instance, with about the same degree of partial justification, $\dot{\gamma} = [\int_\Omega (\dot{\varepsilon}_N^2 + \dot{\varepsilon}_T^2)\, d\Omega]^{1/2}$, or $R(\dot{\gamma}) = \int_\Omega \operatorname{asinh}(\dot{\varepsilon}/c_{R1})\, d\Omega$ with $\dot{\varepsilon} = |\vec{\dot{\varepsilon}}_N|$. Also, different $\dot{\gamma}$ might be appropriate for normal and shear components on the microplanes. Practical differences, though, are small because only the order of magnitude of $R(\dot{\gamma})$ really maters.

EFFECT OF CREEP

If the rate dependence of crack opening is taken into account, the creep (or viscoelasticity) of concrete must be taken into account too. It is natural to do so at the microplane level as well. The creep should be considered as linearly viscoelastic, despite the fact that the stresses exceed 50% of the strength of concrete. This is justified by the recent conclusion (Bažant 1993, 1995) that the nonlinear creep at high stresses is nothing but a manifestation of the rate dependence of the opening of microcracks and, under very high confinement, also rate-dependent plasticity or friction, both of which are modeled separately.

The creep is fully characterized by the compliance function $J(t, t')$, representing the strain at time t caused by a unit uniaxial stress applied at time t', and by the creep Poisson ratio, which can be taken as constant. The constitutive law, based on the principle of superposition, consists of a matrix integral equation over the stress tensor history. For the analysis of impact and penetration, as well as earthquake and short-time loads lasting less than a day, the aging of concrete may of course be neglected. In that case, $J(t, t') \approx J(t - t')$ = function of the time lag $t - t'$. Computationally it is more effective to approximate the integral-type constitutive law with a Kelvin chain or Maxwell chain. This leads to a system of first-order matrix differential equations in time for the partial strains or stresses of the chain, whose values from the preceding time step must be stored.

Since longtime creep is not of interest here, the entire Maxwell chain is not needed. Therefore (in similarity to the approach of Ožbolt and Bažant 1992), the compliance function may be approximated for the duration of loading (e.g., the impact event) by a single spring-dashpot Maxwell rheologic model [Fig. 1(d)]. For uniaxial stress, this model is characterized by the following stress-strain relation:

$$\dot{\varepsilon} = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta} \tag{10}$$

where E = elastic modulus and η = viscosity.

For an explicit analysis, the approximation of this differential equation by forward or backward finite-difference formulae is known to lead to an integration algorithm for structural analysis that is stable only for a sufficiently short time step Δt. Such conditional stability is often inconvenient. As shown for the nonaging viscoelasticity of polymers by Zienkiewicz et al. (1968) and by Taylor et al. (1970), and for aging viscoelasticity of concrete by Bažant (1971), an unconditionally stable algorithm can be obtained by exact integration of the differential equation under the assumption that the strain rate is constant during each time step and varies only by jumps at the beginning of each time step.

So we assume that, during each time step Δt, the strain rate $\dot{\varepsilon}$ is constant. Under that assumption, the solution of (10) within the time step beginning at time t_i is

$$\sigma(t) = \eta\dot{\varepsilon} + (\sigma_i - \eta\dot{\varepsilon})e^{-E(t-t_i)/\eta} \tag{11}$$

Setting $t = t_i + \Delta t$ and $\dot{\varepsilon} \approx \Delta\varepsilon/\Delta t$, this may be rearranged into the following quasielastic incremental stress-strain relation:

$$\Delta\sigma = E''\Delta\varepsilon - \Delta\sigma'' \tag{12}$$

with the following notations:

$$E'' = \frac{1 - e^{-\Delta t/\tau_1}}{\Delta t/\tau_1} E; \quad \Delta\sigma'' = (1 - e^{-\Delta t/\tau_1})\sigma \tag{13a,b}$$

where σ is the value at the beginning of the time step Δt, and $\tau_1 = \eta/E$ = relaxation time of the Maxwell model. Note that, in computer calculations, an overflow may occur in the foregoing expressions when $\xi = \Delta t/\tau_1$ is very small; the first terms of the Taylor series expansions of these expressions must then be used to calculate E'' and $\Delta\sigma''$. Formulae (13) are of the type known generally as the exponential algorithm for linear viscoelasticity (RILEM 1988).

Now the question is what is the optimum approximation of the compliance function $J(t, t')$ by the Maxwell model for the duration of the impact event. As the optimality condition, we may choose the compliance function of the Maxwell model to be tangent to $J(t, t')$ [Fig. 1(d)] at the time $t_0 + t_{ch}$, where t_0 = age of concrete at the time of impact and t_{ch} is a suitably chosen characteristic time. The integration of (10) at time t_{ch} for constant stress $\sigma = 1$ applied at time t_0 gives $\varepsilon = (1/E) + (t_{ch}/\eta)$ and $\dot{\varepsilon} = 1/\eta$. The tangency condition is $J(t_0 + t_{ch}) = \varepsilon$ and $\dot{J}(t_0 + t_{ch}) = 1/\eta$ where $\dot{J}(t, t') = \partial J(t, t')/\partial t$. So, the actual compliance curve $J(t, t')$ and the compliance curve of the Maxwell model are made tangent at time $t_0 + t_{ch}$ by setting

$$E = \frac{1}{J_{ch} - t_{ch}\dot{J}_{ch}}; \quad \eta = \frac{1}{\dot{J}_{ch}} \qquad (14a,b)$$

where $J_{ch} = J(t_0 + t_{ch})$ and $\dot{J}_{ch} = \dot{J}(t_0 + t_{ch})$. As for the characteristic time, it is appropriate to choose $t_{ch} \approx t_{\text{load}}/2$ where t_{load} is the load duration of interest (e.g., the duration of an impact event or earthquake).

The creep Poisson ratio of concrete is approximately constant (being about 0.18). This is fortunate because it means that the volumetric and deviatoric compliance functions are proportional to the uniaxial one, $J(t, t')$, and further, that the Maxwell-type stress-strain relations for all the microplane stress and strain components have proportional viscosities and elastic constants. Thus, according to the approximation (14) by the Maxwell model, the equation preceding (2) (in step 5 of the algorithm) in Caner and Bažant (2000) [or (13*a*) on p. 248 of Bažant et al. (1996a), which reads $\sigma_D^e = \sigma_D^i + E_D(\varepsilon_D - \varepsilon_D^i)$], needs to be replaced by

$$\sigma_D^{ve} = \sigma_D^i + E_D''(\varepsilon_D - \varepsilon_D^i) - \Delta\sigma_D'' \qquad (15)$$

Here subscript D denotes the deviatoric components of the microplane normal stress or strain; superscript i denotes the initial value in the time step; subscript ve denotes the viscoelastic stress increment, which is a generalization of the elastic stress increment and characterizes the response away from the stress-strain boundaries; and the quantities labeled by $''$ are calculated according to the prescription in (13), that is

$$E_D'' = \frac{1 - e^{-\Delta t/\tau_1}}{\Delta t/\tau_1} E_D; \quad \Delta\sigma_D'' = (1 - e^{-\Delta t/\tau_1})\sigma_D^i \qquad (16a,b)$$

Similarly, the remaining equations in steps 4 and 9 of the algorithm in Caner and Bažant (2000) [or in (13*a*) and (14*a*,*b*) in Bažant et al. (1996b)] need to be replaced by

$$\sigma_V^{ve} = \sigma_V^i + E_V''(\varepsilon_V - \varepsilon_V^i) - \Delta\sigma_V'' \qquad (17)$$

$$\sigma_M^{ve} = \sigma_M^i + E_M''(\varepsilon_M - \varepsilon_M^i) - \Delta\sigma_M'' \qquad (18)$$

$$\sigma_L^{ve} = \sigma_L^i + E_L''(\varepsilon_L - \varepsilon_L^i) - \Delta\sigma_L'' \qquad (19)$$

The equation for σ_N in step 6 [or (14*b*) of Bažant et al. (1996a)] is an equilibrium equation that keeps the same form, that is, $\sigma_N^{ve} = \sigma_V^{ve} + \sigma_D^{ve}$, and so does (17) of that paper, ensuing from the principle of virtual work.

A more accurate representation of creep applicable to many orders of magnitude of creep duration could be based on the Maxwell chain model or the Kelvin chain model, with the algorithm described in RILEM (1988) or Bažant (1995). But in that case it would be necessary to introduce for each integration point of each finite element additional internal variables, corresponding to the partial stresses or strains in these models. Their values would have to be stored, and the programming would be more involved.

It might seem that consideration of creep (or viscoelasticity) is unimportant for the extremely short duration of impact. However, experimental data on the nonlinear triaxial stress-strain relations of concrete at the loading rates of impact do not exist and seem next to impossible to acquire. Thus, the knowledge of creep is essential for correlating the behavior under impact to the existing experimental data basis, obtained at the usual strain rates of static laboratory tests, and in particular for determining the effective elastic modulus (dynamic modulus) for the given loading rate (the dynamic modulus for the strain rates of impact can be as much as 50% higher than that in static laboratory tests, the difference being due to creep). The short-time creep must be considered to properly distinguish among the material properties applicable at the different strain rates prevailing for earthquake, blast, ground-shock, and impact loadings.

NUMERICAL IMPLEMENTATION IN WAVE CODE "EPIC"

The present rate-dependent version of microplane model M4 has been implemented at WES in the hydrocode EPIC. Various questions of numerical implementation and algorithms had to be resolved for this purpose.

The rate effect is lagged by one time step, that is, the rate factor for the current step is based on the previously converged step, in order to achieve efficient explicit step-by-step calculations. But this is not the only reason. Because the strains are accumulated when the stress increments are prescribed, the strain rate to be entered as the argument of the rate function may change abruptly from one iteration to the next. This may cause spikes in the response and may result in convergence loss for small enough increment sizes. The lagging of the rate factor allows the boundaries to be kept fixed throughout the iterations in each loading step, which helps convergence. As the increment size is reduced, the error due to the lagging of the rate of loading is decreased.

From the programming viewpoint, the lagging of the rate effect by one step means that there are two more double-precision variables to store, along with all the other stored history variables for every integration point of every finite element. This requires an approximately 2.35% increase in the allocated memory with respect to the model M4 without rate effects.

NUMERICAL SIMULATION AND CALIBRATION BY TEST DATA

In fitting the data, the creep effect can be calibrated separately. This is done by matching the data for the initial elastic modulus at different rates. Then coefficients c_{R1} and c_{R2} may be identified by matching the peak and postpeak behavior for uniaxial stress at different strain rates. For lack of material test data, c_{R0} had to be chosen, and it has been chosen as $c_{R0} = 1$. All the available test data fits were optimized under the restriction that c_{R1} and c_{R2} be the same for all the data. The result was $c_{R1} = 10^{-6}$/s and $c_{R2} = 0.011$.

For lack of creep data, model B3 (Bažant and Baweja 1995a,b,c, 1999) was used to predict all the creep properties other than the overall creep multiplier q_1 from that model, which is determined in this case by optimally fitting the initial elastic moduli of experimental data containing rate-of-loading effects. For this purpose, the following typical parameters were assumed for the B3 model: age of concrete at loading, $t_0 = 28$ days; aggregate-cement ratio by weight, $a/c = 7$; specific cement content, $c = 300$ kg/m^3; and water-cement ratio by weight, $w/c = 0.50$. The characteristic times for rates of loading $\dot{\varepsilon} = 0.2$/s, $3.3 \cdot 10^{-3}$/s, $3.3 \cdot 10^{-5}$/s were determined to be t_{ch} = 0.015 s, 0.9 s, and 90 s, respectively. The values given for t_{ch} approximately correspond to half of the loading duration for each experiment. The optimum value of creep coefficient q_1, $q_1^{\text{opt}} = 21$ MPa^{-1} is found to give the best initial elastic moduli for different rates of loading.

The experimental data on the rate-of-loading effects on concrete are not abundant in the literature. Moreover, all the data are incomplete for the purpose of calibrating a model such as M4. The most complete data set in the literature is probably that of Dilger (1978).

Fig. 2(a) shows the fits of the complete stress-strain curves in uniaxial compression obtained, for different strain rates, by Dilger et al. (1978). The specimen loaded at the highest strain rate failed just after reaching the peak load. The solid lines are the simulations and the data points the measured values. The delocalization of data was not necessary in this case.

The inertial and wave propagation effects were neglected in data fitting because the shortest test duration, which was 0.03 s, was 10 times longer than the travel time of elastic waves across the specimen, which should have sufficed for the wave energy to get dissipated. Figs. 2(b and c), in which the duration is about 0.003 s, serve merely to demonstrate the model performance; no data exist for this rate, but if they did, the inertia effects would have to be taken into account in fitting. The fit shown in Fig. 2(a) indicates that the model is capable of representing the loading rate effects well.

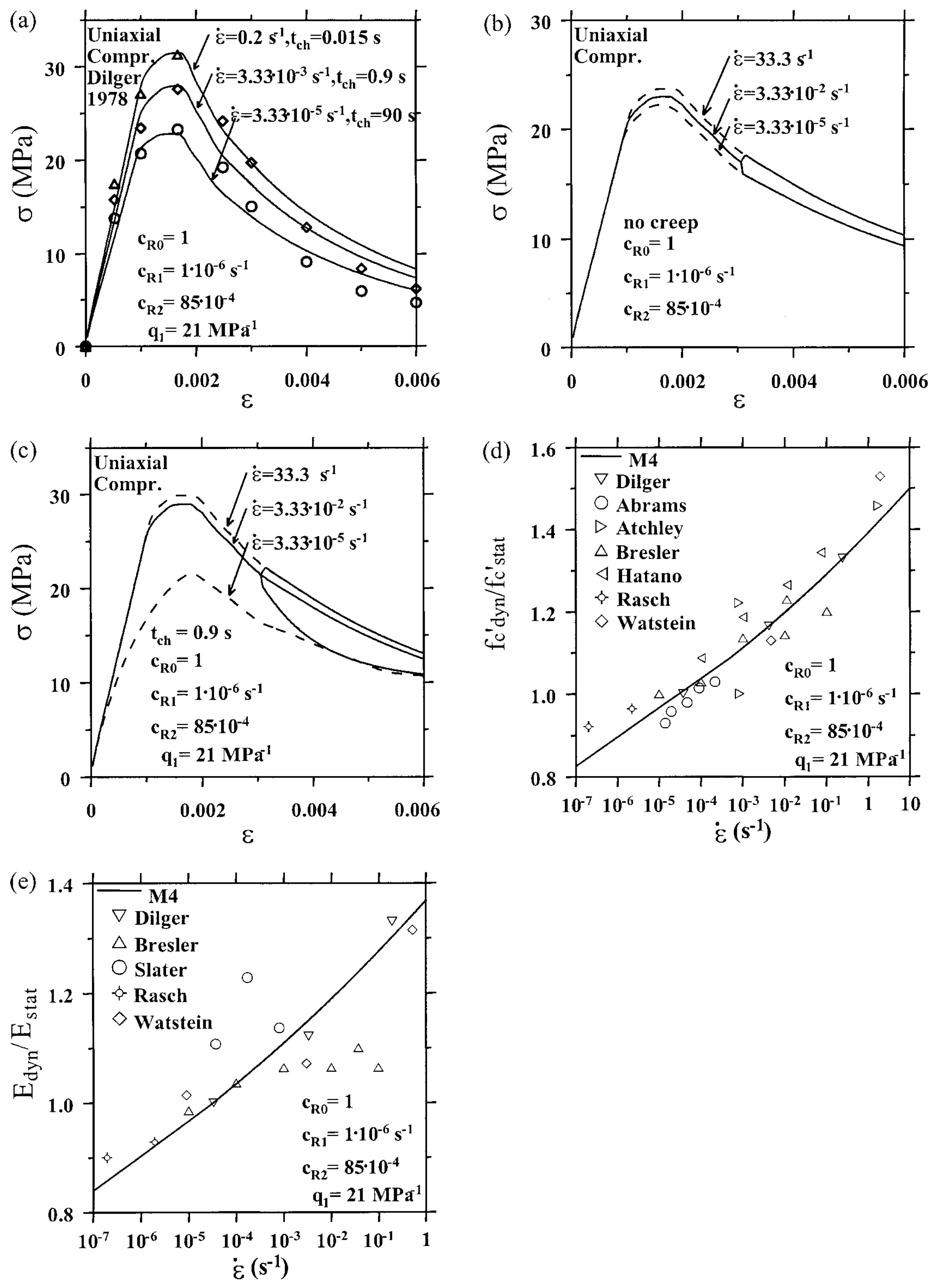

FIG. 2. (a) Fit of Stress-Strain Curves Measured by Dilger (1978) at Various Strain Rates; (b and c) Simulation of Effect of a Sudden Change of Strain Rate, Taking Fracturing Rate into Account, but with or without Effect of Creep; (d) Fit of Main Data from Literature on Effect of Strain Rate on Compression Strength f'_c and on Secant Elastic Modulus E (E = 27.6 GPa, k_1 = 0.000104; k_2 = 200; k_3 = 10; k_4 = 150 Was Used)

Figs. 2(d and e) show the experimental data obtained by various researchers for the effect of strain rate on the compression strength $f'_{c_{\mathrm{dyn}}}$ and on the apparent (dynamic) Young's modulus E_{dyn} in compression, normalized by the values $f'_{c_{\mathrm{stat}}}$ and E_{stat} corresponding to the typical ''static'' loading rate of laboratory tests. Corrections for strain localization were deemed unnecessary for these data. The curves of model M4 are seen to fit well.

Fig. 2(b) shows how a sudden 1,000-fold change of the loading rate affects the postpeak tensile response of model M4. This effect is an important check on the capability of the model to simulate the behavior of the fracture process zone. Qualitatively, the curves in the figure are in good agreement with the recent tests of notched fracture specimens by Bažant et al. (1995) and Tandon et al. (1995), which revealed that a sudden increase of the loading rate in the postpeak softening range reverses softening to hardening. The rehardening is then followed by a second peak and a rapid approach to the response (dashed) curve that would apply if the new rate were constant from the beginning. A sudden decrease of the loading rate was found in these tests to cause a temporary stress relaxation with an almost vertical downward slope. The predictions of model M4 agree well with this behavior. Fits of actual data, however, are not attempted in Fig. 2(b) because they would require nonlocal finite-element analysis with model M4, which is beyond the scope of this paper.

Previous simulation of the behavior in Fig. 2(b) by a rate-dependent cohesive crack model (Bažant and Li 1997) showed that creep (or viscoelasticity) alone gives no reversal of softening to hardening and thus cannot explain the effect of a sudden increase of the loading rate. To explain it, one needs to describe the dependence of the crack-bridging cohesive stresses on the crack-opening rate according to the activation energy theory. These test results and their previous analysis with the cohesive crack model indicate that viscoelasticity of the material alone would be an insufficient hypothesis, and the activation energy theory for fracturing must be introduced. On the other hand, the fracturing rate effect alone suffices to describe the data on the sudden change of loading rate. This is demonstrated by the simulations in Fig. 2(c), in which the creep effect is omitted. As seen, the predicted slope change after a sudden change of loading rate is still realistic. But without creep, there is no effect on the initial apparent elastic modulus (initial slope of the curves).

It would make no sense to simulate the response to a 1,000-fold change of loading rate of the opposite case in which the fracturing rate effect is removed and the creep is the only source of rate effect. The reason is that the approximation of the compliance function by a single Maxwell unit can capture the creep effects only within one order of magnitude of loading rates.

CONCLUSIONS

1. Two types of rate effect in nonlinear triaxial behavior of concrete may be distinguished: (1) Rate dependence of fracturing (growth of microcracks), which is caused by the rate-dependent process of bond ruptures at crack tips and is controlled by activation energy; and (2) linear creep (or viscoelasticity), which occurs in the bulk of the material.
2. Both types of rate dependence are easily implemented in microplane model M4. This makes model M4 suitable for analyzing dynamic problems of concrete structures, including impact, blast, and earthquake effects.
3. Model B3 for predicting concrete creep can be used to characterize short-time creep effects, manifested for instance in the difference between the static and dynamic moduli of elasticity. For the purposes of dynamic loading and short-time static loading, the creep can be approximated by a nonaging Maxwell spring-dashpot model whose response at constant stress is tangent to the compliance function of model B3 for a time-delay characteristic of the structural response.
4. An effective explicit algorithm for step-by-step finite-element analysis is formulated. Good convergence is achieved even for sudden changes of the loading rate.
5. In the case of a sudden increase of strain rate in the postpeak regime, the model gives a reversal of softening to hardening, as required by recent test data. This feature is caused not by creep but by the rate dependence of fracturing. On the other hand, the rate dependence of the initial stiffness and the unloading stiffness is caused not by the rate dependence of postpeak fracturing, but by creep. The peak stress and postpeak softening are affected by both phenomena.
6. Good agreement with the experimentally observed rate effect on concrete is achieved.

APPENDIX I. ALGORITHM FOR EVALUATING *E* AND η FROM MODEL B3

For dynamics and short-time static loading, the drying creep is negligible, and only the basic creep (creep at no moisture exchange) needs to be considered. In that case, one needs to specify for model B3 [Bažant and Baweja (1995a,b,c,) and (improved version) (1999)] the following parameters: Age of concrete at loading t_0; average standard cylindrical compression strength f_c; specific cement content c; water-cement ratio w/c by weight; and aggregate-cement ratio a/c by weight. Further, one must set the characteristic time t_{ch}, using $t_{ch} \approx t_{\mathrm{load}}/2$, and choose the time increment Δt. Also note that the time unit in the following formulae of model B3 is 1 day, and so a conversion from seconds to days will be necessary for parameters t_{ch} and Δt. Otherwise, SI units (MPa) are used.

1. Compute $E_{28} = 4{,}734\sqrt{f_c}$ in MPa, $q_2 = 185.4\sqrt{c}\bar{f}_c^{-0.9}$ in MPa^{-1}, and $q_4 = 20.3(a/c)^{-0.7}$ in MPa^{-1}. Compute $q_1 = 0.6 \cdot 10^6/E_{28}$ in MPa^{-1} and $q_3 = 0.29(w/c)^4 q_2$ in MPa^{-1}; (however, it is much more realistic to carry out a short-time test of given concrete in order to update q_1 and q_2). Set $t = t_0 + 2t_{ch}$ in days. Then compute $r = 1.7t_0^{0.12} + 8$; $Z = t_0^{-0.5}\ln(1 + (t - t_0)^{0.1})$; and $Q_f = (0.086t_0^{2/9} + 1.21t_0^{4/9})^{-1}$. Calculate integral Q of model B3, either by numerical integration by interpolation from the table given by Bažant and Baweja (1995a,b,c, 1999), or more simply but less accurately from the empirical approximation $Q \approx Q_f Z/(Z^r + Q_f^r)^{1/r}$.
2. Compute the compliance function for the characteristic time, $J = q_1 + q_2 Q + q_3\ln(1 + (t - t_0)^{0.1}) + q_4\ln(t/t_0)$ in MPa^{-1}. But if short-time test data on the initial effective elastic modulus of the given concrete at different loading rates are available, calculate the ratios q_2/q_1, q_3/q_1, and q_4/q_1. Then calculate $J = q_1^{\mathrm{opt}}[1 + (q_2/q_1)Q + (q_3/q_1)\ln(1 + (t - t_0)^{0.1}) + (q_4/q_1)\ln(t/t_0)]$, where q_1^{opt} in MPa^{-1} is the optimal value that fits the test data the best [if extensive short-time creep data for the given concrete are available, then it is possible to update both q_1 and q_2, by linear regression, in which case only the ratios q_3/q_2 and q_4/q_2 are based on model B3; see Bažant and Baweja (1995b, 1999)]. In applications to dynamics, in which $t_{ch} << 1$ day, the logarithmic terms are negligible and may be omitted, and one may set at the outset $q_3 \approx 0$ and $q_4 \approx 0$.
3. Similarly, compute $\dot{J} = q_1^{\mathrm{opt}}[0.1((q_2/q_1)t^{-0.5} + q_3/q_1)/(t - t_0 + (t - t_0)^{0.9}) + (q_4/q_1)/t]$ in $(\mathrm{MPa\ day})^{-1}$ and $E_p = 10^6/[J - t_{ch}\dot{J}]$, and evaluate $\eta = 10^6/\dot{J}$, and $\tau_1 = \eta/E_p$.
4. For finite-element analysis: (a) If $\Delta t/\tau_1 \geq 100$, set $C_0 = 0$; otherwise, calculate $C_0 = \exp(-\Delta t/\tau_1)$, and if $\Delta t/\tau_1 \leq 0.001$, calculate $C_1 = 1 - \Delta t/(2\tau_1)$; or, otherwise, cal-

culate $C_1 = (1 - C_0)/(\Delta t/\tau_1)$. These approximations are necessary to avoid overflow in (16), but double-precision arithmetic is still needed in (16), unless the limits on $\Delta t/\tau_1$ were narrowed and more than two terms of Taylor series expansion were used. (b) Compute the new value of effective Young's modulus, $E = C_1 E_p$, for the current time step. (c) Calculate $\sigma_X^{i+1} = E\Delta\varepsilon_X + C_0\sigma_X^i$, where subscript X stands for D, V, M, or L and superscript i is the time index.

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