# FRACTURE MECHANICS OF ASR IN CONCRETES WITH WASTE GLASS PARTICLES OF DIFFERENT SIZES

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**ABSTRACT:** Using waste glass as an aggregate in concrete can cause severe damage because of the alkali-silica reaction (ASR) between the alkali in the cement paste and the silica in the glass. Recent accelerated 2-week tests, conducted according to ASTM C 1260, revealed that the damage to concrete caused by expansion of the ASR gel, which is manifested by strength reduction, depends in these tests strongly on the size of the glass particles. As the particle size decreases, the tensile strength first also decreases, which is expected because of the surface-to-volume ratio of the particles, and thus their chemical reactivity increases. However, there exists a certain worst (pessimum) size below which any further decrease of particle size improves the strength, and the damage becomes virtually nonexistent if the particles are small enough. The volume dilatation due to ASR is maximum for the pessimum particle size and decreases with a further decrease of size. These experimental findings seem contrary to intuition. This paper proposes a micromechanical fracture theory that explains the reversal of particle size effect in the accelerated 2-week test by two opposing mechanisms: (1) The extent of chemical reaction as a function of surface area, which causes the strength to decrease with a decreasing particle size; and (2) the size effect of the cracks produced by expansion of the ASR gel, which causes the opposite. The pessimum size, which is about 1.5 mm, corresponds to the case where the effects of both mechanisms are balanced. For smaller sizes the second mechanism prevails, and for sizes <0.15 mm no adverse effects are detectable. Extrapolation of the accelerated test (ASTM C 1260) to real structures and full lifetimes will require coupling the present model with the modeling of the reaction kinetics and diffusion processes involved.

## INTRODUCTION

The alkali-silica reaction (ASR) is the chemical reaction between the alkalies in portland cement paste and certain kinds of silica found in a variety of natural concrete aggregates. The reaction of silica with alkali ions leads to a reaction product in the form of ASR gel, which expands in time (Glasser 1992; Swamy 1992a,b). The expansion may cause extensive cracking of the cement matrix, thereby reducing the strength of concrete and its useful life (Poole 1992; Swamy 1992). This problem has been known for some 60 years (Stanton 1940; Blanks 1941; Meissner 1941; Vivian 1951; Swenson 1957), but research efforts have been hampered by the long times needed for the reaction to take place. An accelerated standard test of 2-week duration (ASTM C 1260), which recently became available, permits studying the potential reactivity of various aggregates and development of effective ASR-reducing measures.

Various types of glass, including that used for bottles and common household containers, cause some of the worst ASR expansion if used as an aggregate in concrete. Although glass is chemically very similar to regular sand, the chemical bonds of silica in the amorphous structure of glass are not as strong as in the crystalline structure of typical sand. Therefore, most kinds of regular sand do not cause ASR.

The study of ASR in concrete with glass aggregate offers two important benefits. First, the recycling of mixed-color waste glass poses major problems for municipalities everywhere. This problem can be greatly reduced by using crushed waste glass as an aggregate in concrete, assuming the damage caused by ASR could be avoided. Second, because of its relatively simple chemical composition, glass lends itself to an easy study of ASR, its deleterious effects, and mitigating measures.

Two different aspects of mathematical modeling of ASR in concrete may be distinguished: (1) The mechanics of fracture that affects volume expansion and causes loss of strength, with possible disintegration of the material; and (2) the kinetics of the chemical reactions and diffusion processes involved. The basic idea of the first aspect, the fracture aspect, was briefly outlined at two recent conferences (Bažant et al. 1998a,b) [also Bažant (personal communication, 1997)], and its refinement and detailed development are the objective of this paper. A comprehensive mathematical model addressing the second aspect is presented in a follow-up paper by Bažant and Steffens (2000). The present study will be limited to the ASR effects at one fixed time of observation, corresponding to the 2-week duration of the recently introduced accelerated test (ASTM C 1260).

Recent experiments at Columbia University, New York (Meyer and Baxter 1998; Jin 1998) have demonstrated that the ASR-induced expansion measured by the ASTM test, as well as the compression strength, depends strongly on the size of the glass particles in the concrete mix. As expected, it was found that, for large particles, a decrease of particle size causes an increase of volume expansion and damage due to ASR. Surprisingly, however, this proved to be true only for particle sizes down to a certain ‘‘pessimum’’ (worst) size, which was about 1.5 mm. When the particle size was further reduced below this pessimum, the opposite was observed—the expansion decreased and the strength increased with a further reduction of particle size, and for particle sizes below about 0.15 mm the observed expansion was the same as for specimens with no glass.

The objective of the present paper is to explain these startling results theoretically by fracture mechanics alone, leaving aside the kinetics of chemical reactions and diffusion processes. A ‘‘bare-bones’’ simplification of the chemical kinetics and diffusion aspects of the problem will be introduced, to keep the fracture analysis simple and to achieve an easily understandable analytical solution.

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## VOLUME OF REACTION PRODUCT

The volume $V_a$ of glass that has undergone ASR, per unit volume of concrete, depends on a number of variables: $V_a = F(V_g, D, p, T, w, t)$, in which $V_g$ = total volume of glass aggregate per unit volume of concrete (volume fraction); $D$ = diameter of glass particles; $p$ = pressure on the surface of the glass particle; $T$ = temperature; $w$ = specific water content in the pores per m$^3$ of concrete; and $t$ = time. Because we are interested only in the volume $V_a$ after the lapse of a certain fixed time $t$ (2 weeks in the case of the ASTM test), the dependence of $F$ on $t$ need not be considered. As a simplification, it seems reasonable to neglect the variables $w$, $T$, and $p$. Thus, it will be assumed that

$$\text{for} \quad D/2 > R_{cr}: \quad V_a = V_r; \quad \text{for} \quad D/2 \le R_{cr} \quad V_a = V_g \qquad (1a,b)$$

in which

$$V_r = (\pi/6)[D^3 - (D - 2R_{cr})^3]\bar{N}, \quad \bar{N} = V_g/(\pi D^3/6) \qquad (2)$$

Here $\bar{N}$ = number of glass particles per unit volume of concrete; $V_r$ = volume of the layers (spherical shells) that have undergone ASR in all the particles per unit volume of concrete; and $R_{cr}$ = constant = critical radius = radius of the largest spherical glass particle that will get fully penetrated by ASR within the fixed duration of the accelerated ASTM test ($R_{cr}$ is determined by water diffusion to the reaction front and the kinetics of the chemical reaction). Substitution of (2) into (1) provides

$$\text{for} \quad D/2 > R_{cr}: \quad V_a(D) = \left[1 - \left(1 - \frac{2R_{cr}}{D}\right)^3\right] V_g \qquad (3a)$$

$$\text{for} \quad D/2 \le R_{cr}: \quad V_a(D) = V_g \qquad (3b)$$

## STRESS INTENSITY FACTOR IN INITIAL AND TERMINAL STAGES

For the sake of simplicity, we consider a regular array of cubical cells, each containing one spherical glass particle embedded in the matrix of hardened cement mortar. The number of glass particles per unit volume of concrete is $\bar{N} = V_g/(\pi D^3/6)$, and for a cubic array of particles $\bar{N} = 1/s^3$, where $s$ = particle spacing. Thus, $D = (6V_g/\pi)^{1/3}s$. The term $V_g$ is assumed to be constant, which means that the cells with glass particles of various sizes are geometrically similar (i.e., $D/s$ = constant). In numerical calculation, the case $V_g$ = 10% will be assumed, for which $D/s = 0.58$.

The ASR proceeds from the surface of the particle inward. The expanding layer of the solid ASR gel at the surface of the particle applies radial pressure $p$ on the surrounding cement mortar. The pressure causes cracks to initiate from preexisting flaws at the surface of the particle, as shown in Fig. 1 (left). For the sake of simplicity, the crack planes in all the cells are assumed to be parallel, and then each periodically repetitive cell represents the overall material behavior. The ASR gel also penetrates into the cracks and applies pressure $p$ on their surfaces.

In view of symmetry, the analysis may be restricted to a single cubical cell containing a spherical glass particle in the center [Fig. 1 (right)]. The cement mortar is separated from the glass by a growing layer of ASR gel, which provides no bond. As a simple idealized geometry, one may consider an initially circular ring crack to grow from the spherical void containing the glass particle [Fig. 2(a)].

To figure out the stress intensity factor $K_I$ due to $p$, it is helpful to realize that the stress field caused by pressure $p$ on the crack and void surfaces is a superposition of the stress field caused by the volumetric (hydrostatic) stress $p$ applied by the gel onto the void and crack surfaces and the stress field caused by externally applied tensile stress $\sigma = -p$ [Fig. 2(b)]. The former field is a homogeneous field of hydrostatic pressure $p$, for which $K_I = 0$, and so $K_I$ due to $p$ alone is the same as $K_I$ due to $\sigma = -p$. Hence, what determines $K_I$ is not $\sigma$ and $p$ individually but solely the sum $\sigma + p$, where $\sigma$ = externally applied tensile stress and $p$ = hydrostatic stress.

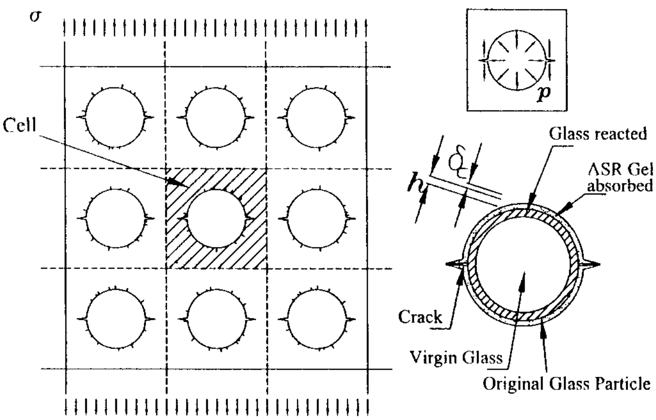

FIG. 1. (Left) Idealized Cubic Array of Expanding Spherical Glass Particles Surrounded by Flaws and Dominant Growing Cracks Filled by Pressurized ASR Gel; (Right Top) One Cell with Glass Particle and Crack; (Right Bottom) Layer of Reacted Glass (ASR Gel Layer Growing into Glass Particle) and Layer of ASR Gel Forced into Capillary Pores of Concrete

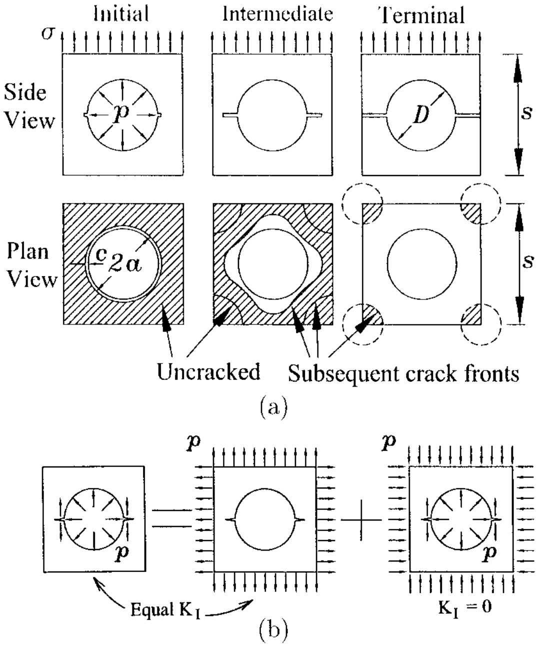

FIG. 2. (a) Idealized Evolution of Crack Front from Initial Stage of Shallow Annular Cracks Around Glass Particle to Terminal Stage of Small Circular Uncracked Ligaments; (b) Superposition Argument Revealing Equivalence of Crack Pressurization and Stress Externally Applied on One Cell

The $K_I$ caused by the normal applied stress components parallel to the crack plane can be neglected (it is exactly zero in the limit $D/c \to 0$). This means that the $K_I$ due to pressure $p$ applied on the crack and void surfaces is nearly the same as the $K_I$ due to uniaxial applied stress $\sigma$ normal to the crack plane. The applied stress $\sigma$ is understood as the average stress acting on the planes between the adjacent cells.

Let the crack growth be characterized by the dimensionless damage variable $\omega = A_c/s^2$, where $A_c$ = area of the crack and the spherical void in one cubical cell of size $s$. Then, for reasons of dimensionality, the stress intensity factor must have the form

$$K_I = (\sigma + p)\sqrt{s}k(\omega) \quad (4)$$

where $k(\omega)$ = dimensionless function.

The initial value of $\omega$ before the ring crack starts to grow is $\omega_0 = \pi D^2/4s^2$. In the initial phase of crack growth, in which there is a roughly circular ring crack around the void, $A_c = \pi(D + 2c)^2/4$. Consequently, $c = s\sqrt{\omega/\pi} - D/2$ for small $\omega - \omega_0$. The stress intensity factor $K_I$ of the ring crack must initially be the same as $K_I$ for a surface crack of an elastic half-space, since the influence of the spherical void on the crack is negligible, and thus it can be considered as an edge crack, when the crack is small (Baratta 1978). So, according to a well-known solution (Tada et al. 1985), $K_I = 1.12(\sigma + p)\sqrt{\pi c}$ or

$$k(\omega) = 1.12\sqrt{\pi}\sqrt{\sqrt{\omega/\pi} - D/2s} \quad (\text{for small } \omega - \omega_0) \quad (5)$$

Thus, the dimensionless stress intensity factor $k$ is treated here as a function of $\omega$, which in turn is a function of the equivalent annular crack length $c$.

When $\omega$ becomes larger, the shape of the crack front ceases to be circular. In principle it is governed by the condition of maximum energy release, but in practice this problem is very complex. However, the shape of the crack must again become very simple when $\omega \rightarrow 1$. This fact may be exploited for estimating function $k(\omega)$ for the intermediate values of $\omega$ using the philosophy of asymptotic matching.

At the end of the fracture process ($\omega \rightarrow 1$), the remaining uncracked ligaments in a cross section of the material may be considered to get localized into circular ligaments of radius $r$ centered at the corners of each cubic cell. When $r << s$, $K_I$ at the circular edge of these ligaments must be the same as $K_I$ of a circular ligament joining two elastic half-spaces (Fig. 2), which is $K_I = P/2\sqrt{\pi r^3}$, where $P = (\sigma + p)s^2$ = force resultant normal to each ligament (Tada et al. 1985). Because, in this terminal stage of fracture, $\omega = 1 - \pi(r/s)^2$, one has $r = (s/\sqrt{\pi})\sqrt{1 - \omega}$ for small $1 - \omega$. Consequently

$$k(\omega) = \frac{1}{2}\pi^{1/4}(1 - \omega)^{-3/4} \quad (\text{for small } 1 - \omega) \quad (6)$$

The 3D fracture evolution between the initial and terminal stages is doubtlessly extremely complicated, but it need not be represented accurately. It should suffice to find a smooth formula for $k(\omega)$, which matches (5) and (6) asymptotically for very small and very large values of $\delta\omega = \omega - \omega_0$. The plots of $K_I$ as a function of $\omega$ according to (5) and (6) are shown as the dashed curves in Fig. 3. The question now is how to describe a smooth transition between these plots; in other words, how to find an asymptotic matching formula.

The following formula, plotted as the solid curves in Fig. 3 has been found to possess the desired asymptotic properties:

$$k^2(\omega) = 1.12^2\pi\left(\sqrt{\frac{\omega}{\pi}} - \frac{D}{2s}\right) + \frac{\sqrt{\pi}}{4(1 - \omega)^{3/2}} - \frac{\sqrt{\pi}}{4(1 - \omega_0)^{3/2}} \cdot\left(1 + \frac{3}{2}\,\frac{\omega - \omega_0}{1 - \omega_0}\right) + k_2(\omega - \omega_0)^2 + k_3(\omega - \omega_0)^3 + \cdots \quad (7)$$

where $k_2, k_3, \ldots$ are arbitrary constants. Because there is no way to determine them, it will be assumed that $k_2 = k_3 = \cdots = 0$. To verify the asymptotic properties, one may check that the values of $k^2(\omega)$ and its first derivative at $\omega = \omega_0$ match those of (5), and that the values of $k^{-2}(\omega)$ and its first derivative at $\omega = 1$ match those of the inverse of (6).

Fig. 3 also shows, as a dotted curve, function $k(\omega)$ for the case of a circular (penny-shaped) crack of radius $a = c + D/2$ in an infinite half-space, for which $K_I = (2/\pi)(\sigma + p)\sqrt{\pi a}$. It might be thought that such a crack should be a good approximation for the intermediate stage of fracture evolution and that (7) should be adapted to match this curve in the middle. However, as seen from Fig. 3, this curve does not match at all the smooth interpolation between the asymptotic cases shown as the dashed curves. Therefore, the solution for a circular crack is not helpful for improving the present asymptotic matching formula.

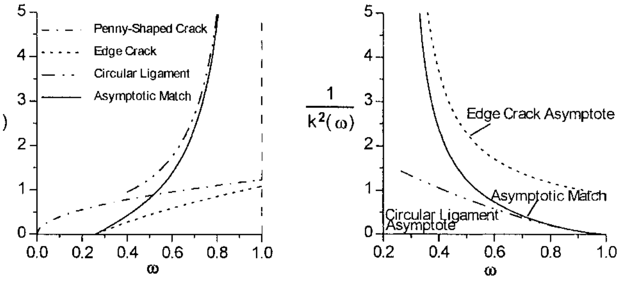

**FIG. 3. Asymptotic Matching Formula (7) for $k(\omega)^2$ (Left) and $1/k(\omega)^2$ (Right) $k(\omega)$ as Function of Cracked Area Fraction $\omega$ (Damage Parameter), Pictured for Glass Volume Fraction 10%, with Initial Asymptote of Shallow Edge Crack, Final Asymptote of Small Circular Ligament, and Formula for Circular (Penny-Shaped) Crack; $k(\omega)$ = Dimensionless Stress Intensity Factor**

One might wonder why the fracture is not modeled by a single circular (penny-shaped) crack in an infinite space, as done in the preceding studies (Bažant et al. 1998a,b). This simpler model, however, has a much more limited applicability. It applies only to (1) the case of very low glass volume fractions $V_g$, for which the particles are spaced so widely that the cracks emanating from the adjacent particles cannot significantly interact and the final stages of material breakup (tearing of small remaining ligaments) need not be modeled; and (2) the case of high $V_g$ (close spacing) if the cracks develop at only some of the particles and if those particles that are surrounded by cracks are spaced widely. But for $V_g$ = 10% the particles are spaced quite closely ($D = 0.58s$). Besides, the case of each particle producing a crack appears more likely to occur and is, of course, the more prudent assumption to make. This is the reason why the simpler approach of Bažant et al. (1998a) has not been adopted.

## MACROSCOPIC DEFORMATION AND VOLUME CHANGE OF MATERIAL

The average macroscopic strain due to fracture in the direction normal to the crack may be written as follows:

$$\varepsilon_{fr} = \frac{\Delta v_{fr}}{s^3} = \frac{1}{s^3}\,C(\omega)(\sigma + p) \quad (8)$$

where $\Delta v_{fr}$ = volume change due to fracture; and $C(\omega)$ = material compliance, which increases with fracture growth and depends on the damage parameter $\omega$. To determine $\omega$, we consider the complementary potential energy per cell, which is $\Pi^* = (1/2)C(\omega)F^2$ where $F = (\sigma + p)$ = stress resultant over the cross section of the cell. Thus, $\Pi^* = (1/2)C(\omega)(\sigma + p)^2$ per cell. The energy release rate per cell is $\mathcal{G} = \partial\Pi^*/\partial A_c$ or $\mathcal{G} = (1/s^2)[\partial\Pi^*/\partial\omega] = (1/2)(\sigma + p)^2[\partial C(\omega)/\partial\omega]$, where $\sigma + p$ = constant.

According to the famous Irwin's relation [e.g., Bažant and Planas (1998)], $\mathcal{G} = K_I^2/E'$, where $E' = E/(1 - \nu^2)$; $E$ = Young's modulus; and $\nu$ = Poisson's ratio of cement mortar. Substituting this into the last equation and integrating over $\omega$, with the initial value $C(\omega_0)$, one obtains

$$C(\omega) = \frac{s^3}{E'} + \frac{2s^3}{E'}\int_{\omega 0}^{\omega} k^2(\omega')\,d\omega' \quad (9)$$

The advantage of the asymptotic matching formula (7) is that this integral can be easily evaluated. The result is

$$C(\omega) = \frac{s^3}{E'} + \frac{2s^3}{E'}\,\phi(\omega) \qquad (10)$$

in which $\phi(\omega) = \bar{\phi}(\omega) - \bar{\phi}(\omega_0)$; and $\bar{\phi}(\omega)$ is

$$\bar{\phi}(\omega) = \frac{\sqrt{\pi}}{2}\frac{1}{\sqrt{1-\omega}} + 1.12^2\pi\left(\frac{2}{3}\frac{\omega^{3/2}}{\sqrt{\pi}} - \frac{D}{2s}\,\omega\right) - \frac{\sqrt{\pi}}{4}\frac{\omega}{(1-\omega_0)^{3/2}}\left\{1 + \frac{(3\omega - 1.5\omega_0)}{1-\omega_0}\right\} \qquad (11)$$

The average macroscopic strain of the fractured material then follows from (8).

The complementary energy per cell is the work of $p$ and may also be expressed as $\Pi^* = (1/2)p\Delta v_{fr}$ where $\Delta v_{fr}$ = volume change of the crack (with void) caused by uniform pressure $p$ applied on its surfaces. Comparison of this expression for $\Pi^*$ to that expressed in terms of the average strain due to $p$ [i.e., $\Pi^* = (1/2s^3)p\delta\varepsilon_{fr}$], implies that $\delta v_{fr} = s^3\delta\varepsilon_{fr}$. Therefore, the relative volume change due to fracture is

$$\Delta v_{fr} = C(\omega)(\sigma + p) \qquad (12)$$

## CONDITION OF VOLUME COMPATIBILITY

The following volume compatibility condition must obviously be satisfied:

$$\Delta v_{fr} = \langle \Delta v_{gel} - \Delta v_{cap} \rangle \qquad (13)$$

where $\langle\ \rangle$ (positive part of argument) is defined by $\langle X \rangle = \max(X, 0)$; $\Delta v_{cap}$ = volume of the ASR gel (per cell) that diffuses within a given time (duration of test) into the capillary pores in the cement mortar and into the crack; and $\Delta v_{gel}$ = volume (per cell) of all the ASR gel under pressure $p$ minus the volume of glass from which the gel has formed

$$\Delta v_{gel} = s^3 V_a \lambda_a (1 - C_a p) - s^3 V_a \qquad (14)$$

Here $\lambda_a$ = ratio of the specific volume of the ASR gel at zero pressure $p$ to the specific volume of glass from which is has formed; and $C_a$ = bulk compressibility of the ASR gel, per unit volume of gel (which is probably only slightly less than the bulk compressibility of water).

Around the circular particle, the ASR gel under pressure may be assumed to diffuse within a fixed time into the cement mortar to a uniform depth $\delta_c$, as shown in Fig. 1 (right bottom). Along the cracks, the depth of penetration of diffusion may be assumed to decrease roughly linearly with the distance from the particle and be zero at the crack tip. Large cracks that do not have a circular front may be replaced, for the purpose of estimating $\Delta v_{cap}$, by equivalent circular cracks of radius $a_{ef}$ having the same area (i.e., $\pi a_{ef}^2 = s^2\omega$, which implies that $a_{ef} = s\sqrt{\omega/\pi}$). Thus, neglecting the dependence of $\Delta v_{cap}$ on $p$, one has

$$\Delta v_{cap} = n_c\left\{\frac{\pi}{6}[(D + 2\delta_c)^3 - D^3] + \delta_c\langle l \rangle 2\pi\left(a_{ef} - \frac{2}{3}\,l\right)\right\} \qquad (15)$$

where $n_c$ = capillary porosity of cement mortar; and $l = a_{ef} - (D/2 + \delta_c)$; $\langle\ \rangle$ is used to deal with cracks shorter than $\delta_c$.

Eq. (13) can be equivalently and more conveniently written as follows:

$$\Delta v_{fr} + \Delta v_{cap} + \Delta v_0 + C_a(\Delta v_{\mathrm{ASR}} - \Delta v_{cap})p = \Delta v_{\mathrm{ASR}} \qquad (16)$$

in which

$$0 \le \Delta v_{cap} \le \Delta v_{\mathrm{ASR}} - \Delta v_0 \qquad (17)$$

where $\Delta v_{\mathrm{ASR}} = s^3 V_a \lambda_a$ and $\Delta v_0 = s^3 V_a$. The left-hand side of (16) represents the space available to the ASR gel, and the right-hand side is the volume of the gel generated. The positive part signs $\langle\ \rangle$ in (13) can be removed by limiting $\Delta v_{cap}$ as in (17).

## EFFECT OF ASR ON MATERIAL STRENGTH

When the strength limit is reached, the crack must be growing. This means that $K_I$ must be equal to its critical value, the fracture toughness $K_c$; therefore

$$\sigma + p = \frac{K_c}{\sqrt{s}k(\omega)} \qquad (18)$$

This value may be substituted into (4); $p$ may then be solved from (16) and (18) and substituted into (18). This yields the stress at failure (i.e, the strength) $f'_{t,D}$ as a function of $\omega$ and $D$

$$f'_{t,D} = \sigma = \frac{K_c}{\sqrt{s}k(\omega)}\left\{1 + \frac{C(\omega)}{C_a(\Delta v_{\mathrm{ASR}} - \Delta v_{cap})}\right\} - \frac{\{\Delta v_{\mathrm{ASR}} - \Delta v_0 - \Delta v_{cap}\}}{C_a(\Delta v_{\mathrm{ASR}} - \Delta v_{cap})} \qquad (19)$$

The value of $\omega$, however, remains undetermined. An additional condition is needed.

If the curve of $\sigma$ versus $\omega$ exhibited a peak, the maximum load (or stability limit under dead load) would be determined by the condition $\partial\sigma/\partial\omega = 0$. However, according to (19), $\sigma$ monotonically decreases with $\omega$. Therefore the material failure must take place as soon as the crack becomes critical (i.e., as soon as $K_I = K_c$), as proposed already by Griffith (1921). He noted that the determination of material strength from fracture mechanics makes sense only if one considers inevitable flaws (preexisting microcracks).

To determine the effective flaw size that gives the correct value of strength, one must consider the case of concrete containing no glass particles, whose tensile strength is denoted as $f_t^0$. One may consider each flaw to behave approximately as a circular (penny-shaped) crack of radius $c$ in an infinite solid under uniaxial stress $\sigma$ normal to the crack. In that case $K_I = 2\sigma\sqrt{c/\pi}$. Setting $K_I$ and $K_c$ and $\sigma = f_t^0$, one obtains the critical radius of an effective circular flaw, $c_{cr} = \pi K_c^2/4f_t^{0^2}$.

It is reasonable to assume that a critical flaw of size $2c_{cr}$, with $c_{cr} = \psi D$, $\psi$ = constant, is encountered at the surface of the glass particle. As an approximation, we thus consider that there exists an initial ring crack whose front has the radius $a_{cr} + D/2 + c_{cr}$. The material strength limit is reached when $K_I$ of this crack attains the critical value $K_c$.

Based on the known radius of the critical ring crack, the expression (19) assures the dependence of the material strength on the glass particle diameter $D$.

There must be a certain minimum size $D_a$ below which the usual mechanism of failure of concrete without glass takes over. Furthermore, for $D \to \infty$, (19) predicts $f'_t \to 0$, which is also unreasonable. This means that a certain residual strength $f'_0$ governed by a plastic mechanism that is insensitive to particle size must be added to the expansion of (19). So it appears that (19) should be generalized as

$$f'_{t,D} = \sigma = \frac{K_c}{\sqrt{s}k(\omega)}\left\{1 + \frac{C(\omega)}{C_a(\Delta v_{\mathrm{ASR}} - \Delta v_{cap})}\right\} - \frac{\{\Delta v_{\mathrm{ASR}} - \Delta v_0 - \Delta v_{cap}\}}{C_a(\Delta v_{\mathrm{ASR}} - \Delta v_{cap})} + f'_0 \qquad (20)$$

This equation, as already mentioned, cannot apply for very small $D$. When the effect of diminishing particle size $D$ is studied and $D$ becomes smaller than $c_{cr}$, the assumption that each particle in the cubical array has a ring crack becomes meaningless because the diameter of the front of such a crack would be greater than the spacing $s$ ($s/D$ being constant). Therefore, (20) must be corrected; $f'_{t,D} = \min\{f'_t, \text{Eq. (20)}\}$.

The uniaxial compressive strength can then be roughly estimated from the American Concrete Institute (ACI) empirical relation $f'_t = 6\sqrt{f'_c}$, in which $f'_c$ and $f'_t$ must be given in psi (1 psi = 6,895 Pa). Consequently, in psi, $f'_{c,D} \approx (f'_{t,D}/6)^2$.

## VOLUME EXPANSION CAUSED BY ASR

If $\sigma < f'_{t,D}$, the material can be stable even for a ring crack radius larger than $a_{cr}$. The expansion due to ASR is the volumetric strain caused by pressure $p$ with $\sigma = 0$. Thus, from (8) and (16), the volumetric strain caused by ASR depends on $D$ as follows:

$$\varepsilon_{fr} = \frac{C(\omega)}{s^3}\,\frac{\Delta v_{\mathrm{ASR}} - \Delta v_0 - \Delta v_{cap}}{C(\omega) + C_a(\Delta v_{\mathrm{ASR}} - \Delta v_{cap})} \tag{21}$$

## COMPARISON TO TEST RESULTS

The literature apparently contains no experimental information on the bulk compressibility $C_a$ of the ASR gel. But it is known that the ASR gel swells because of imbibing water, and so the swollen gel must contain much water. Therefore, its $C_a$ ought to be of the same order of magnitude as the bulk compressibility $C_w$ of liquid water ($C_a \approx C_w$), which is far higher than that of undamaged concrete—about 20× higher at room temperature and atmospheric pressure (and still much higher that that of the cement gel without capillary pores). Hence, by comparison, we may neglect all the other elastic compressibilities except those due to crack.

Note that (20) bears some similarity to the Petch formula (also known as the Hall-Petch formula) for the effect of size of crystals on the strength of polycrystalline metals (Cottrell 1964; Tetelman and McEvily 1967), which has been discussed in Bažant and Kim (1985) in relation to the dependence of tensile strength of concrete on the maximum aggregate size.

## COMPARISON WITH EXPERIMENTS AND DISCUSSION OF RESULTS

Figs. 4 and 5 show the results of the recent experiments at Columbia University (Jin 1998; Meyer and Baxter 1998), which demonstrated that the ASR-induced expansion, as measured by the 14-day accelerated ASTM C 1260 test, depends strongly on the size of the glass particles in the concrete mix. Fig. 4 shows the expansion of mortar bars in which 10% of the total graded sand was replaced with clear soda-lime (bottle) glass of various particle diameters $D$ (assuming that the sand constitutes about 85% of the total volume of mortar, one has $V_g \approx 0.085$). If the glass is ground to standard mesh size number 50 or finer ($D$ = 0.02 in. = 0.5 mm), the measured as well as calculated 14-day expansions are equal to or less than those of reference bars that contain no glass but a slightly reactive sand from Long Island.

The values of the parameters in the equations obtained cannot be predicted but must be obtained by fitting test results. Once the parameter values are determined, the theory can of course be used to predict other situations.

Formulas (20) and (21) with (17) have been optimally fitted to the test results of Meyer and Baxter (1998) and Jin (1998). The fits are shown by the curves in Figs. 4 and 5. The dashed and dash-dot lines in Fig. 4 show the asymptotic trends according to the present theory.

Eq. (21) for the expansion fits the observed trend very well (Fig. 4). The optimum values of the parameters are $\lambda = 1.34$, $n_c = 0.31$, $R_{cr} = 0.0086$ in., $\delta_c = 0.0050$ in., $E'\lambda C_a = 0.23$, and $\psi = 0.095$, where $\psi$ = ratio of $c_{cr}$ to $D$.

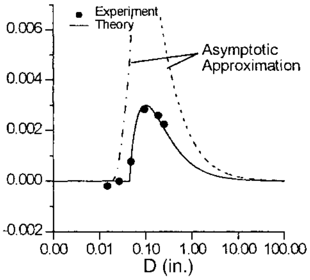

**FIG. 4. Measured Dependence of Volume Expansion due to ASR on Diameter *D* of Glass Particles and Optimum Fit by Eq. (21)**

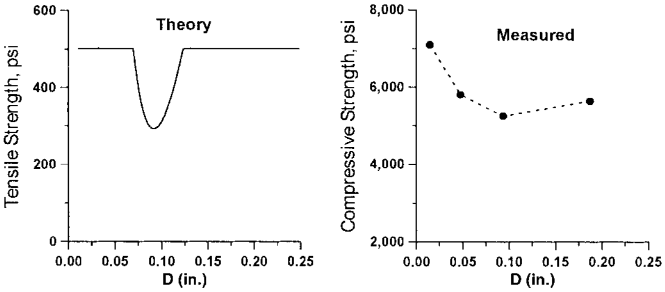

**FIG. 5. Measured Dependence of Compression Strength on Diameter *D* of Glass Particles, as Reported by Meyer and Baxter (1998) and Jin (1998) and Tensile Strength According to Eq. (20) with Eq. (1)**

Eq. (20) for the tensile strength $f'_t$ cannot be compared to Meyers' test data because what was measured was the compressive strength $f'_c$. As is well known, prediction of the compression strength of concrete by fracture mechanics is a formidable problem, much harder than the prediction of tensile strength. Nevertheless, one relation is simple: When $f'_c$ increases (or decreases), $f'_t$ does the same. In this sense, the predictions of (20) for $f'_t$ do agree with the measured $f'_c$ values and, most importantly, predict correctly the pessimum particle size, as can be seen by comparing the plots in Fig. 5.

One could convert the measured $f'_c$ to $f'_t$ using the approximate ACI relation $f'_t = \sqrt{f'_c} \times$ constant. This would obviously indicate the variation of $f'_t$ with $D$ to be spread over a much wider range of $D$ than predicted by the present theory. However, the ACI relation is doubtless too crude for this purpose. Certainly, it is not intended for capturing the differences between the effects of crack size on the tensile and compression strengths. To check the present theory it would be necessary to measure the tensile strength itself.

## INTUITIVE EXPLANATION OF PARTICLE SIZE EFFECT

The intuitive reasons for the reversals of the dependence on $D$ may be summarized as follows:

1. For strength:
    - At large enough $D$, two effects compete: (a) According to (3), $V_a(D) \approx 6V_gR_{cr}/D \propto 1/D$, which alone would cause the strength to vary as $D$; while (b) the effect of crack size on the strength is roughly $1/\sqrt{a_{ef}}$ or $1/\sqrt{D}$. The former effect on strength wins because, approximately, $\Delta f'_t = D/\sqrt{D} = \sqrt{D}$.
    - At small enough $D$, on the other hand, according to (3), $V_a = V_g$ = constant (as all the glass has reacted). This wipes out the former effect and only the latter effect remains, so that, approximately for small enough $D$, $\Delta f'_t \propto 1/\sqrt{D}$.

2. For expansion:
   - At large enough $D$, the crack opening stiffness is negligible, and so the expansion is almost equal to the volume increment of the gel. Since the fraction of reacted glass diminishes with increasing $D$, the mass of gel produced diminishes as well and so does the expansion.
   - At small enough $D$, on the other hand, the reason that the expansion due to ASR decreases with a decreasing particle size $D$ is that $V_a = V_g$ = constant while the crack opening stiffness rapidly increases with a decreasing crack size and thus also with decreasing $D$.

In other words, the trends for sufficiently small and sufficiently large particle sizes are opposite because the volume fraction of the glass converted to the ASR gel during the accelerated test increases with a decreasing particle size if the particle are not too small but ceases to decrease when the particles are so small that all of the glass is converted into gel. Thus, the effect of particle size on the gel surface-to-volume ratio and volume fraction overpowers the fracture mechanics effect only for large enough sizes.

The foregoing reasoning is based exclusively on fracture mechanics. It does not take into account the possibility that, for extremely small particles, most of the ASR with the resulting expansion may take place already before taking the first measurement, as specified by ASTM C 1260.

The effect of $V_a$ is essentially the effect of the specific internal surface $S_g$ of glass particles. When the particles are relatively large, only a relatively thin surface layer of the glass particle gets reacted during the 2 week ASTM test while the core of the glass particle remains shielded from ASR. In that case, the amount of ASR gel produced depends mainly on the exposed surface of particles per unit volume of concrete, which increases with decreasing particle size, roughly in inverse proportion to $D$. When, however, the particles become smaller than a certain critical (pessimum) size, all of their volume gets reacted during the 2 week test. This then means that the surface area of the particles becomes irrelevant. A further decrease of the particle size cannot increase the mass of the ASR gel produced, and so the volume fraction of ASR becomes independent of particle size.

According to this explanation, the pessimum size should of course increase if the test duration increases. However, the effect of test duration is more complicated.

An increase of test duration allows more gel to be squeezed into the capillary pores. It also affects the relative progress of various diffusion processes involved. The reaction kinetics evolves in time and the reaction affinities (driving forces) depend on the local specific moisture content in the pores of concrete and consequently on the supply of water by macrodiffusion. The stresses produced by the expansion are relaxed by creep as well as by time-dependent crack growth. Understanding the effect of the duration of ASR will require mathematical modeling of the kinetics of the chemical reactions and diffusion processes involved in ASR. This is the subject of a separate study at Northwestern University, Evanston, Ill. (Bažant and Steffens 2000).

According to the present mathematical model, addition of glass particles to the concrete mix could never produce an increase in strength, no matter how small the particles. However, a strength increase may be expected when the glass particles are extremely fine. Similar to the effect of silica fume or various pozzolana in concrete, the solid gel produced by ASR may fill various micropores flaws, and microcracks in concrete, thus reducing stress concentrations in the microstructural lattice.

## CONCLUSIONS

1. The effect of ASR on tensile strength and volume expansion may be analyzed according to fracture mechanics considering a repetitive cubical cell of mortar with one spherical glass particle embedded in it. A ring crack may be assumed to be caused by the expansion.
2. In fracture mechanics of the problem, two simple stages of crack evolution may be discerned. The situation may be assumed to be approximately equivalent to an edge crack in an elastic half-space in the initial stage and to small circular uncracked ligaments in the terminal stage. An asymptotic matching formula may be used to interpolate between the asymptotic cases. The evolution of the stress intensity factor caused by pressure from the gel and by externally applied stress may be conveniently expressed in terms of a damage parameter representing the crack area fraction. The deformation of the unit cell can be integrated analytically on the basis of the stress intensity factor solution.
3. Calculation of the pressure may be based on the condition of compatibility of the volume expansion of the ASR in a given ratio, the change of volume of the crack and spherical void containing the particle, and the volume of the gel that gets squeezed into the adjacent capillary pores.
4. The numerical solution of the problem allows an excellent fit of the observed volume expansions as a function of the particle size. As for the compression strength, the present theory is inadequate for predicting it. If the tensile strength is approximately estimated from the measured compression strength, the agreement of the present theory with tests is only qualitative; however, the simple ACI formula for tensile strength is doubtlessly insufficient for capturing the effects of pressurized cracks. Measurements of tensile strength are needed to check and calibrate the theory.
5. The intuitive reason that, for large enough particles, a decrease of particle size causes a decrease of tensile strength of concrete is that the surface area of all the glass particles per unit volume increases. This means that a larger volume fraction of glass undergoes ASR, producing larger pressures. The reason that, for small enough particles, a decrease of the glass particle size causes the tensile strength to increase is that all of the glass volume that undergoes ASR is independent of particle size (the whole particles having reacted) while at the same time a decrease of particle size causes a reduction of the stress intensity factor (or the energy release rate) at the preexisting inevitable flaws near the surface of the reacted particle. The worst (pessimum) particle size is obtained when the foregoing two effects are balanced.
6. It remains to be seen whether similar conclusions apply for long lifetimes of real structures. To clarify it, the present model will have to be coupled with a model of the reaction kinetics and diffusion processes involved.

## APPENDIX I. MODIFIED SOLUTION FOR LOW GLASS CONTENT

Fig. 4 demonstrates that for relative large ratios $D/s$ such as 0.58, the curve of the stress intensity factor for a circular (penny-shaped) crack cannot dominate at any stage of crack growth. However, when the ratio $D/s$ is very small (low glass content), the stress intensity factor should approach the solution of a circular (penny-shaped) crack, which is $k^2(\omega) = 4\pi^{-3/2}\sqrt{\omega}$, before the solution for the terminal stage of small circular ligaments is approached. In that case, the stress intensity factor formula (7) needs to be modified, which may approximately be done as follows:

$$k^2(\omega) = \min\left\{1.12^2\pi\left(\sqrt{\frac{\omega}{\pi}} - \frac{D}{2s}\right), \frac{4}{\pi^{3/2}}\sqrt{\omega}\right\} + \frac{\sqrt{\pi}}{4(1-\omega)^{3/2}} - \frac{\sqrt{\pi}}{4(1-\omega_0)^{3/2}}\left(1 + \frac{3}{2}\frac{\omega-\omega_0}{1-\omega_0}\right) + k_2(\omega-\omega_0)^2 + k_3(\omega-\omega_0)^3 + \cdots \quad (22)$$

Again, it is possible to integrate function $\phi(\omega)$ analytically. The solution is similar to the solution already presented.

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