

FRACTURE ENERGY RELEASE AND SIZE EFFECT IN BOREHOLE BREAKOUT

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SUMMARY

The paper presents a simple approximate analytical solution of the remote stresses that cause the collapse of a borehole or other circular cylindrical cavity in an infinite elastic space. Regions of parallel equidistant splitting cracks are assumed to form on the sides of the cavity. Their boundary is assumed to be an ellipse of a growing horizontal axis, the other axis remaining equal to the borehole diameter. The slabs of rock between the splitting cracks are assumed to buckle as slender columns, and their post-critical stress is considered as the residual stress in the cracked rock. The buckling of these slab columns is assumed to be resisted not only by their elastic bending stiffness but also shear stresses produced on rough crack faces by relative shear displacements. The energy release from the infinite medium caused by the growth of the elliptical cracking region is evaluated according to Eschelby's theorem. This release is set equal to the energy dissipated by the formation of all the splitting cracks, which is calculated under the assumption of constant fracture energy. This yields the collapse stress as a function of the elastic moduli, fracture energy, ratio of the remote principal stresses, crack shear resistance characteristic and borehole diameter. The collapse stress as a function of crack spacing is found to have a minimum, and the correct crack spacing is determined from this minimum. For small enough diameters, the crack spacing increases as the $(4/5)$ -power of the borehole diameter, while for large enough diameters a constant spacing is approached. In contrast to plastic solutions, the breakout stress exhibits a size effect, such that for small enough diameters the breakout stress decreases as the $(-2/5)$ -power of the borehole diameter, while for large enough diameters a constant limiting value is approached. Finally, some numerical estimates are given and the validity of various simplifying assumptions made is discussed.

1. INTRODUCTION

The sudden catastrophic collapse of boreholes in rock, called the breakout, as well as the collapse (such as rock burst) of various other types of cavities due to high compressive stresses in the rock mass, has been studied extensively and various important results have been obtained.¹⁻²² However, most studies have been based on the theory of plasticity, which does not give a sufficiently realistic description of the inelastic behaviour of rock, except at very high confining

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pressures. Such pressures, however, never exist near the sides of cavities. Cavities usually appear to fail due to fracture of rock, and the failure process is described by fracture mechanics better than plasticity. A fully realistic description would no doubt require a combination of both theories, but the analysis would then become rather complicated.

The most important practical consequence of fracture mechanics is that it predicts size effect, that is, the remote compressive stress that causes a borehole to fail must decrease as the borehole size increases. On the other hand, according to plasticity (or any other failure theory expressed in terms of stress and strain), there is no size effect. But the existence of the size effect has been detected experimentally (e.g. References 19, 23, 24).

Fracture mechanics has so far been well developed only for tensile fractures, and to some extent shear fractures; their microscopic mechanism, however, still usually consists of tensile cracks. Compressive fractures are not very well understood at present, although it is clear that their mechanism involves, in one way or another, some form of tensile cracking depending on the structure geometry. The purpose of this study is to formulate appropriate simplifying assumptions and use fracture mechanics concepts to obtain an analytical solution of borehole collapse that reveals the size effect and is sufficiently simple to be clearly understood—one benefit that numerical solutions cannot provide. The plasticity aspects of failure will have to be neglected to make an analytical solution feasible. The reality may be expected to be somewhere between the solutions of plasticity and fracture mechanics, but probably in most situations much closer to the latter.

2. ENERGY RELEASE DUE TO GROWTH OF ELLIPTICAL CAVITY

Consider a circular cylindrical borehole of radius R and horizontal axis z in an infinite elastic space that is in a state of plane strain (Figure 1a) and is subjected at infinities to uniform compressive stresses $\sigma_{x\infty}$ and $\sigma_{y\infty}$ in the directions of Cartesian co-ordinates x and y . We will assume that failure tends to enlarge the circular cavity into an ellipse of horizontal axis $a \geq R$, with the vertical axis remaining equal to R .

Based on Eshelby's solution of the stress field and using the superposition method, explained in detail for example by Mura,²⁵ one can calculate the loss of the potential energy (per unit thickness in the z -direction) of an infinite, initially uniformly stressed elastic space caused by cutting out an elliptical hole

$$\Delta\Pi_1 = -\frac{\pi}{2E'}[(a+2R)R\sigma_{x\infty}^2 + (2a+R)a\sigma_{y\infty}^2 - 2aR\sigma_{x\infty}\sigma_{y\infty}] \quad (1)$$

where $E' = E/(1-\nu^2)$, E = Young's elastic modulus of the rock, ν = Poisson's ratio, and $\nu' = \nu/(1-\nu)$. $\Delta\Pi_1$ represents the sum of the work of the stresses on the strain changes outside the ellipse, which are non-uniformly distributed and decay with the distance from the ellipse, and the work of the stresses on the strain changes inside the removed elliptical cutout, which are, according to the famous Eshelby's theorem, uniformly distributed within the ellipse.

Equation (1) gives the potential energy change when the stresses within the elliptical region are reduced to zero. Later we will need also the potential energy change $\Delta\Pi'$ when the initial vertical stress $\sigma_{y\infty}$ is reduced to a certain finite critical stress σ_{cr} rather than to zero. In that case the calculations according to Eshelby's theorem yield

$$\Delta\Pi'_1 = -\frac{\pi}{2E'}[(a+2R)R\sigma_{x\infty}^2 + (2a+R)a\sigma_{y\infty}^2 - 2aR\sigma_{x\infty}\sigma_{y\infty} - 2a^2\sigma_{cr}^2] \quad (2)$$

Equation (1) may be checked by considering the limiting case $R \rightarrow 0$, for which the elliptical hole

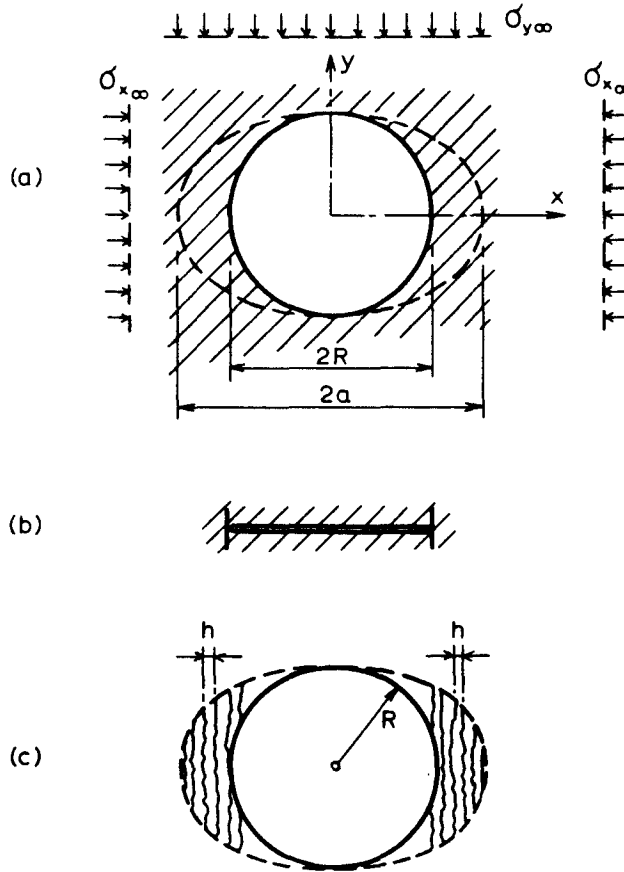


Figure 1. (a) Growth of an elliptical cracking region from a circular borehole, and (b) limit case of a crack

becomes a horizontal crack (Figure 1b). In that case Equation (1) reduces to:

$$\Delta\Pi = -\frac{\pi}{E'}\sigma_{y\infty}^2 a^2 \tag{3}$$

This coincides with the expression for the energy loss of an infinite space due to creating a crack of length $2a$. Indeed, as is well-known (e.g. References 26 and 27), the energy release rate per crack tip is K_1^2/E' where $K_1 = \sigma_{y\infty}\sqrt{\pi a}$ = stress intensity factor, and by integrating one has, for both crack tips combined, $\Delta\Pi = 2\int(K_1^2/E')da$, which is the same result as equation (3).

Proof of equation (2). Consider an infinite elastic body subjected at infinity to a uniform applied stresses σ_∞ , let a uniform eigenstrain ϵ^* be applied to an ellipsoidal domain Ω contained in this infinite body. The values of the eigenstrain ϵ^* are such that the stress is zero everywhere in the ellipsoid after ϵ^* is applied. This means that (if the infinite body is free from any external force) the stress in the ellipsoid induced by ϵ^* will be $-\sigma_\infty$. Because the stress is zero everywhere in the ellipsoid, the ellipsoid can be cut out from the infinite body without affecting its stresses and deformation. Thus, the change of potential energy of the infinite body caused by the applied eigenstrain is the same as the loss of potential energy caused by cutting out the ellipsoid from the

infinite body. This potential energy loss can be calculated as follows (e.g. Reference 25):

$$\Delta\Pi = -\frac{1}{2} \int_{\Omega} \boldsymbol{\sigma}^T \boldsymbol{\varepsilon}^* dV - \int_{\Omega} \boldsymbol{\sigma}_0^T \boldsymbol{\varepsilon}^* dv = -\frac{V}{2} \boldsymbol{\sigma}^T \boldsymbol{\varepsilon}^* - V \boldsymbol{\sigma}_0^T \boldsymbol{\varepsilon}^* \quad (4)$$

where V is the volume of the ellipsoid; $\boldsymbol{\sigma}_0$ and $\boldsymbol{\sigma}$ are the stress vectors in the ellipsoid when the infinite body is subjected to external forces alone or eigenstrain $\boldsymbol{\varepsilon}^*$ alone, respectively.

If plane-strain cases are considered, the ellipsoid becomes an elliptic cylinder and the relationship between eigenstrain $\boldsymbol{\varepsilon}^*$ and the stress induced in the elliptic cylinder is:²⁵

$$\begin{aligned} \frac{\mu}{1-\nu} \left[-2 + \frac{R^2 + 2aR}{(a+R)^2} + \frac{R}{a+R} \right] \varepsilon_x^* + \frac{\mu}{1-\nu} \left[\frac{R^2}{(a+R)^2} - \frac{R}{a+R} \right] \varepsilon_y^* \\ - \frac{2\mu\nu}{1-\nu} \frac{a}{a+R} \varepsilon_z^* = \sigma_x \end{aligned} \quad (5)$$

$$\begin{aligned} \frac{\mu}{1-\nu} \left[-2 + \frac{a^2 + 2aR}{(a+R)^2} + \frac{a}{a+R} \right] \varepsilon_y^* + \frac{\mu}{1-\nu} \left[\frac{a^2}{(a+R)^2} - \frac{a}{a+R} \right] \varepsilon_x^* \\ - \frac{2\mu\nu}{1-\nu} \frac{R}{a+R} \varepsilon_z^* = \sigma_y \end{aligned} \quad (6)$$

$$-\frac{2\mu\nu}{1-\nu} \frac{a}{a+R} \varepsilon_z^* - \frac{2\mu\nu}{1-\nu} \frac{R}{a+R} \varepsilon_y^* - \frac{2\mu}{1-\nu} \varepsilon_z^* = \sigma_z \quad (7)$$

where μ and ν are the Lamé constants; a and R are the axes of the ellipse in the x - and y -directions. If the applied stress components at infinity are $\sigma_{x\infty}$ and $\sigma_{y\infty}$, the stress component in the z -direction is $\nu(\sigma_{x\infty} + \sigma_{y\infty})$. Substituting $-\sigma_{x\infty}$, $-\sigma_{y\infty}$ and $-\nu(\sigma_{x\infty} + \sigma_{y\infty})$ for σ_x , σ_y and σ_z in the above three equations and solving them, we obtain the eigenstrain components

$$\varepsilon_x^* = \frac{(v-1)(a+2R)}{2\mu a} (-\sigma_{x\infty}) + \frac{1-\nu}{2\mu} (-\sigma_{y\infty}) \quad (8)$$

$$\varepsilon_y^* = \frac{1-\nu}{2\mu} (-\sigma_{x\infty}) + \frac{(v-1)(2a+R)}{2\mu R} (-\sigma_{y\infty}) \quad (9)$$

with $\varepsilon_z^* = 0$. The energy loss $\Delta\Pi$ per unit thickness in the z -direction can now be calculated from substituting the above expressions into Equation (A1).

$$\begin{aligned} \Delta\Pi &= -\frac{V}{2} (-\boldsymbol{\sigma}_\infty^T) \boldsymbol{\varepsilon}^* - V \boldsymbol{\sigma}_\infty^T \boldsymbol{\varepsilon}^* = -\frac{V}{2} \boldsymbol{\sigma}_\infty^T \boldsymbol{\varepsilon}^* = -\frac{\pi a R}{2} (\sigma_{x\infty} \varepsilon_x^* + \sigma_{y\infty} \varepsilon_y^*) \\ &= -\frac{\pi}{2E'} [(a+2R)\sigma_{x\infty}^2 + (2a+R)a\sigma_{y\infty}^2 - 2aR\sigma_{x\infty}\sigma_{y\infty}] \end{aligned} \quad (10)$$

where $E = 2(1+\nu)\mu$ and $E' = E/(1-\nu^2)$.

Now consider the loss of potential energy when a uniformly stressed infinite body is cut by an elliptic cylinder whose surface tractions along the surface of the elliptic hole corresponding to the uniform stress state $\sigma_y = \sigma_{cr}$, with other components being zero. The loss of potential energy for this case is expressed similarly to equation (4), except that one term must be added as follows:

$$\Delta\Pi = -\frac{V}{2} \boldsymbol{\sigma}^T \boldsymbol{\varepsilon}^* - V \boldsymbol{\sigma}_0^T \boldsymbol{\varepsilon}^* - \frac{V}{2} (\boldsymbol{\sigma} + \boldsymbol{\sigma}_0^T) \boldsymbol{e} \quad (11)$$

Here the last term represents the elastic strain energy stored in the elliptic cylinder when both the external forces at infinity are applied and the eigenstrain ϵ^* occurs; e is the elastic strain vector in the elliptic cylinder. Substitution of the expressions for σ , σ_0 , ϵ^* and e into equation (11) finally yields

$$\Delta\Pi = -\frac{V}{2}[-\sigma_{x\infty}\epsilon_x^* + (-\sigma_{y\infty} + \sigma_{cr})\epsilon_y^*] - V[\sigma_{x\infty}\epsilon_x^* + \sigma_{y\infty}\epsilon_y^*] - \frac{V}{2}\sigma_{cr}\frac{\sigma_{cr}}{E'} \quad (12)$$

Substitution of equations (8) and (9) then proves equation (2). QED

Equation (1) gives the potential energy change from the case of no cavity to the case of an elliptical cavity. By setting in equation (1) $a = R$, we get the potential energy change from the case of no cavity to the case of circular cavity of radius R (Figure 1a):

$$\Delta\Pi_0 = -\frac{\pi R^2}{2E'}(3\sigma_{x\infty}^2 + 3\sigma_{y\infty}^2 - 2\sigma_{x\infty}\sigma_{y\infty}) \quad (13)$$

Subtracting equation (13) from equation (2), we obtain the potential energy change when the stress in the regions between the original circle and the circumscribed ellipse is reduced from $\sigma_{y\infty}$ to σ_{cr}

$$\begin{aligned} \Delta\Pi = \Delta\Pi_1 - \Delta\Pi_0 = & -\frac{\pi}{2E'}[(aR - R^2)\sigma_{x\infty}^2 + (2a^2 + Ra - 3R^2)\sigma_{y\infty}^2 \\ & + 2R(R - a)\sigma_{x\infty}\sigma_{y\infty} - 2a^2\sigma_{cr}^2] \end{aligned} \quad (14)$$

3. RESIDUAL STRAIN ENERGY AFTER COMPRESSION FRACTURING

If $\sigma_{cr} = 0$, the foregoing expression includes the release of all the strain energy originally stored in the zone between the ellipse and the original circle (Figure 1c). However, it is a particular feature of compression fracturing that this zone cannot be assumed unloaded to zero stress. Compression fracture in quasibrittle microinhomogeneous materials such as most rocks initiates as a system of parallel, roughly equidistant, splitting cracks having the direction of the minimum principal stress (in our case $\sigma_{y\infty}$). These vertical splitting cracks (which initially cause exfoliation at borehole surface and later extensive slabbing^{23, 24, 28}); may eventually get organized to form inclined bands, equivalent to shear bands (Bažant and Ožbolt²⁹), which in our case may be imagined to form along the contour of the ellipse. This aspect, however, does not seem to be essential for calculating the residual strain energy.

Now what is the mechanism that dictates the residual vertical stress σ_{cr} ? If the spacing of the vertical cracks is relatively small, the stress that can be carried by the thin slabs of the material between the adjacent vertical splitting cracks must obviously be limited by elastic buckling³⁰ (this is a discrete version of the idea proposed, for an elastic continuum weakened by smeared parallel cracks, in Reference 31; see also Section 11.7 in Reference 32). So we will consider that these slabs (Figure 1), of thickness h , buckle in the manner of fixed-end columns of a certain length $2L$, equal to the crack length.

It is now useful to recall the initial post-critical behaviour of a perfect elastic column (e.g. Section 1.9 in the textbook of Bažant and Cedolin³²). The diagram of the axial load P of such a column versus the axial load-point displacement u becomes nearly horizontal upon reaching the critical load, i.e. the Euler bifurcation load (the post-critical slope is still positive, equal to $P_{cr}/2L$, but this is negligible compared to the pre-critical slope); see Figure 2. According to this idea, the

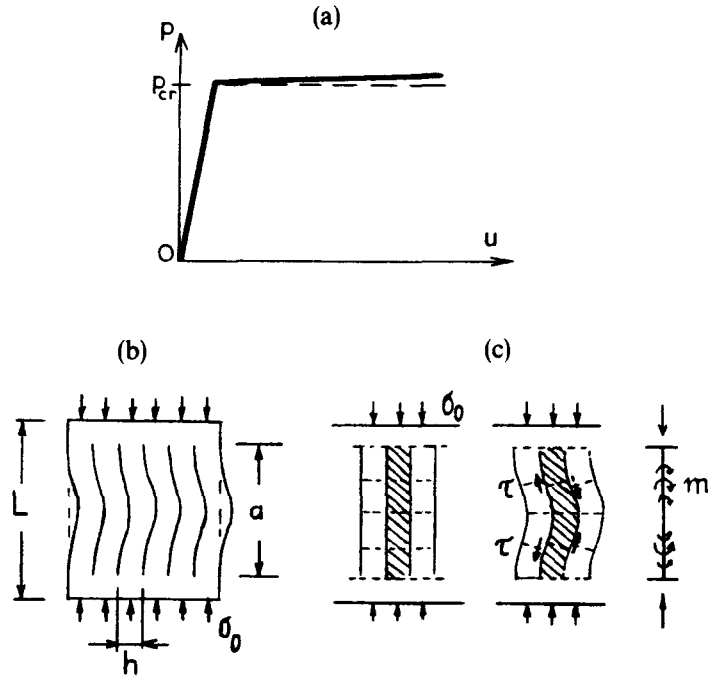


Figure 2. (a) Diagram of axial load vs. axial load-point displacement for column buckling, (b) simultaneous buckling of rock slabs between parallel cracks, and (c) crack shear stresses

vertical compressive stress in the parallel slab-columns of thickness h will not be zero but

$$\sigma_{cr} = -\frac{\pi^2 E' I}{L^2 h} = -\frac{\pi^2 E' h^2}{L^2 12} \quad (15)$$

where $I = h^3/12 =$ centroidal moment of inertia of the cross-section of the slab (per unit thickness in the z -direction), $h =$ spacing of the equidistant splitting cracks, and $L =$ average (effective) half-length of the vertical cracks at the moment of failure.

We will now assume the deformation fields at the moment of failure of small and large boreholes (i.e. the modes of failure) to be geometrically similar, proportional to the borehole radius R . This means we assume that

$$L = k R \quad (16)$$

where $k =$ empirical positive constant < 1 . (This assumption is supported by the following argument: if L were not proportional to R , then we would have $L = kR^n$, where $n =$ constant $\neq 1$; but then, for increasing R , the ratio L/R would tend either to infinity or to zero, that is, the mechanism of collapse would change, which seems irrational.)

In contrast to tensile cracks, the compression splitting cracks have one particular property—their opening displacement is, according to the present model of simultaneously buckling parallel slab-columns (Figure 2b), zero. At the same time, the cracks in rock are rough and transmit shear stresses τ when the opposite faces are subjected to shear. Now, to accommodate the buckling deflections of the adjacent slabs, relative shear displacements Δ between the contacting crack faces must inevitably arise (Figure 2c); $\Delta = w'(y)h$, where $w(y) =$ deflection curve of each

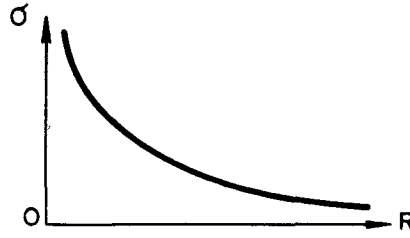


Figure 3. Calculated dependence of remote effective stress causing borehole collapse on the borehole radius

slab-column. The shear stress transmitted across the crack due to surface roughness (Figure 3) may be approximately assumed to be proportional to Δ ; thus, $\tau = G\gamma_{ef}$, where $\gamma_{ef} = \Delta/\lambda = w'h/\lambda$; G = elastic shear modulus of rock and λ = empirical length = material property representing the thickness of an intact rock layer whose elastic shear relative displacement due to unit shear stress is the same at that between the crack faces. The shear stresses acting from both sheared cracks exert on the slab-column a distributed moment $m = rh$ (Figure 2c). The moment differential equation of equilibrium of the slab-column is $M' + Pw' + m' = -V$, where M = bending moment, V = shear force and $P = -\sigma_y h$ = axial compression force. Therefore, the differential equation for the deflection curve is $E'Iw'''' + (P - Gh^2/\lambda)w'' = 0$, where $I = h^2/12$. The lowest critical stress for fixed-end boundary conditions is then easily found to be

$$\sigma_{cr} = -\frac{\pi^2 E' h^2}{12L^2} - \frac{h}{\lambda} G \quad (17)$$

The work of shear stress τ is not included in the strain energy since crack shear is inelastic, irreversible (this work might be included in the dissipated energy expression, but it is negligible at the start of buckling).

The residual strain energy (per unit thickness in the z -direction) contained between the ellipse and the initial circle is given by the bending energy of all the slab-columns, which may now be approximately expressed as

$$\Pi_{cr} = (\pi aR - \pi R^2) \frac{\sigma_{cr}^2}{2E'} = \frac{\pi R(a - R)}{2E'} \left(\frac{\pi^2 E' h^2}{12k^2 R^2} + \frac{h}{\lambda} G \right)^2 \quad (18)$$

4. ENERGY DISSIPATED BY FRACTURING AND ENERGY BALANCE

The energy dissipated by fracturing of the rock is the sum of the energies dissipated by all the vertical splitting cracks, i.e.

$$\Delta W_f = (\pi aR - \pi R^2) \frac{G_f}{h} \quad (19)$$

(per unit thickness in the z -direction), in which G_f/h is the energy dissipated per unit volume of the rock and G_f is the fracture energy of the rock ($G_f = K_{Ic}^2/E'$, where K_{Ic} = fracture toughness of the rock).

The net energy loss due to passing from a circular borehole in intact rock to an elliptical damage zone with vertical splitting cracks now is, instead of equation (14),

$$\Delta \Pi = \Delta \Pi'_1 - \Delta \Pi_0 + \Pi_{cr} \quad (20)$$

The energy balance (principle of conservation of energy) requires that $-\Delta\Pi = \Delta W_f$. Assuming the parallel cracks to form progressively, one after another, we need to differentiate equation (14) with respect to a . Thus, we get the incremental energy balance condition:

$$-\frac{\partial(\Delta\Pi)}{\partial a} = \frac{\partial(W_f)}{\partial a} \quad (21)$$

which yields

$$\frac{\pi}{2E'} [R\sigma_{x\infty}^2 + (4a + R)\sigma_{y\infty}^2 - 2R\sigma_{x\infty}\sigma_{y\infty} - 4a\sigma_{cr}^2] = \frac{\pi R}{2E'} \left(\frac{\pi^2 E' h^2}{12k^2 R^2} + \frac{h}{\lambda} G \right)^2 + \frac{\pi R G_f}{h} \quad (22)$$

We are interested in the start of borehole breakout, which occurs when $a = R$. Substituting this value of a into the last equation, we obtain

$$\sigma_{x\infty}^2 + 5\sigma_{y\infty}^2 - 2\sigma_{x\infty}\sigma_{y\infty} - 4\sigma_{cr}^2 = \left(\frac{\pi^2 E' h^2}{12k^2 R^2} + \frac{h}{\lambda} G \right)^2 + \frac{2E' G_f}{h} \quad (23)$$

Now, introducing the following definition of the effective applied stress:

$$\sigma_{ef} = \sigma_{y\infty} \left(1 - \frac{2\sigma_{x\infty}}{5\sigma_{y\infty}} + \frac{\sigma_{x\infty}^2}{5\sigma_{y\infty}^2} \right)^{1/2} \quad (24)$$

and denoting

$$F(R, h) = \left(\frac{\pi^2 E' h^2}{12k^2 R^2} + \frac{h}{\lambda} G \right)^2 + \frac{2E' G_f}{5h} \quad (25)$$

Equation (13) may now be written simply in the form

$$\sigma_{ef}^2 = F(R, h) \quad (26)$$

where F is a function of R and h .

The question now is how to estimate the spacing h of the vertical splitting cracks. In this regard, it is interesting to note that $F(R, h)$ as a function of h possesses a minimum. From this, a new, simple concept comes to mind.³⁰ The spacing h that will occur is that which minimizes the applied effective stress σ_{ef} . In other words, the splitting cracks will occur at the lowest compressive stress they can (this concept could be proven on the basis of the Gibbs' statement of the second law of thermodynamics in the manner shown in Chapter 10 of the textbook of Bažant and Cedolin).³² The necessary condition of minimum is that

$$\frac{\partial F(R, h)}{\partial h} = 0 \quad (27)$$

After substituting equation (25) for $F(R, h)$ and differentiating, we obtain

$$\frac{5\pi^4 E'^2}{72k^4 R^4} h^5 + \frac{11\pi^2 E' G}{12k^2 R^2 \lambda} h^4 + \frac{5G^2}{\lambda^2} h^3 - E' G_f = 0 \quad (28)$$

This is an algebraic equation of fifth degree for h . Although a numerical solution would be easy, a closed-form solution of h is not possible. However, it will suffice to examine the asymptotic cases.

For sufficiently small R , the terms with h^4 and h^3 become negligible compared to the term with h^5 , and the solution then is

$$h = C_* R^{4/5} \quad C_* = \left(\frac{72k^4 G_f}{5\pi^4 E'} \right)^{1/5} \quad (\text{small } R) \quad (29)$$

From this result³⁰ we see that the spacing of the splitting cracks should increase with the borehole diameter. This property has been observed by Cook³³ and co-workers.

On the other hand, for sufficiently large R , the first two terms of equation (28) may be neglected, and the solution is

$$h = \left(\frac{E' G_f \lambda^2}{5G^2} \right)^{1/3} = \text{constant} \quad (\text{large } R) \quad (30)$$

If we substitute equation (29) into equation (26) and take the asymptotic approximation of equation (26) for small R , we obtain

$$\sigma_{ef} = C_1 R^{-2/5}, \quad C_1 = \left(\frac{\pi^2 5^{3/2}}{48k^2} E'^3 G_f^2 \right)^{1/5} \quad (\text{small } R) \quad (31)$$

while, if we substitute equation (30) into equation (26) and take the asymptotic approximation of equation (26) for large R , we obtain

$$\sigma_{ef} = 3 \left(\frac{GE' G_f}{5\lambda} \right)^{1/3} = C_0 = \text{constant strength} \quad (\text{large } R) \quad (32)$$

For the intermediate values of R we cannot get a closed-form expression. However, the following combination of equations (31) and (32) has the right asymptotic properties for both small and large R and is probably a good approximation that should suffice for practical purposes:

$$\sigma_{ef} \approx C_1 R^{-2/5} + C_0 \quad (33)$$

5. DISCUSSION OF SIZE EFFECT AND NUMERICAL ESTIMATES

Equation (19) indicates that there is a size effect, which is understood as the dependence of the nominal stress at failure (nominal strength, in our case coincident with σ_{ef}) on the size—in our case the borehole radius R , provided that geometrically similar situations are compared. A basic property of plasticity, as well as all other theories with failure criteria expressed in terms of stress and strain tensors, is that there is no size effect (see e.g. Reference 32, Chapters 12 and 13; and References 34–36). Linear elastic fracture mechanics (LEFM) exhibits in general the strongest possible (deterministic) size effect—the nominal strength decreases as $\text{size}^{-1/2}$.

Since the foregoing analysis used LEFM, it is, thus, interesting to realize why the size exponent in equation (31) is $-2/5$ rather than $-1/2$. The reason is that, instead of localizing into a single dominant crack, fracture has been assumed to be distributed over a large zone with an area proportional to the diameter of the borehole. If we assumed a single splitting crack with a length proportional to the borehole diameter, the exponent in equation (31) would have come out as $-1/2$. On the other hand, the exponent would have come out as 0 (i.e. we would have no size effect even for small R) if we assumed the crack spacing h to be the same for every borehole diameter, with the cracking zone area proportional to the diameter (in this case the energy dissipation due to fracture per unit volume would be constant, independent of the borehole diameter, same as in plasticity). It is because of the theory of elastic buckling (and because L increases with R) that we found the crack spacing to increase with the borehole diameter less than proportionally. It is for this reason that the size effect exponent in equation (31) has come out to be intermediate between $-1/2$ and 0, that is, intermediate between the exponents for single-crack LEFM and for plasticity.

The fact that the compressive stress that causes borehole breakout exhibits a size effect has been observed experimentally^{6,23} and has been also predicted by finite element models of non-local type, for example models with couple stresses.^{9,37}

The foregoing analysis tacitly implied the assumption that the failure mode of the borehole is symmetric. Based on the experience with certain other fracture problems (Reference 32, Section 12.5 and Reference 38), one may expect that the loading path might exhibit a bifurcation, after which the failure process proceeds along a non-symmetric secondary path, corresponding to a borehole collapsing non-symmetrically, only on one side of the cavity. Unfortunately, the non-symmetric collapse mode does not seem amenable to a simple analytical solution. The present symmetric solution should represent an upper bound on the actual critical stress for collapse. It may also be pointed out that the symmetric and non-symmetric response paths probably give the same type of size effect and dependence on other basic parameters. Thus, it may well be possible to use the present solution at least qualitatively, even if the actual collapse is non-symmetric.

Another important simplification has been our use of LEFM. The fracture of rock, of course, shows significant departures from LEFM (e.g. References 35 and 39). This may be approximately described by assuming the energy release rate required for fracture growth to be variable (rather than being equal to constant G_I) and to increase with the crack length a according to a given function $R(a)$ called the R-curve (resistance curve). If an increasing R-curve were introduced into the present type of analysis, the resulting size effect would become weaker. However, measurements of the R-curve for the present type of situation are lacking. It is debatable whether any increase of $R(a)$ is appropriate at all when many parallel closely spaced cracks propagate simultaneously, or when the cracks are much longer than the size of the inhomogeneities in rock.

Related to possible R-curve behaviour, the splitting cracks in rock may be discontinuous, capable of transmitting some reduced transverse tensile stresses as well as shear stresses. Capability of shear stress transmission must further arise from the fact that these cracks are no doubt rather tortuous, permitting interlock of the asperities opposing relative slip of the crack surfaces which must take place during buckling. These properties, which have been neglected, would increase the value of σ_{cr} .

A further simplification has been the geometry of the cracking region. Experimental observations of borehole breakout show that the cracking regions on the sides of the borehole tend to have a roughly triangular shape and generally a smaller height than the length of the vertical cross-sections of the ellipse (Figure 4). But for such geometry a simple analytical solution would probably be impossible. Moreover, implicit to our assumption of an elliptical cracking region has been the hypothesis that the cracking regions for boreholes of different diameters are geometrically similar and their size is proportional to the borehole diameter. If the ratio of the average length of the splitting cracks to the borehole diameter decreased with increasing borehole

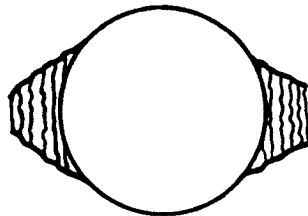


Figure 4. More realistic shape of cracking regions on the sides of a borehole

diameter (i.e. the cracking localized), then the size effect would be stronger than we have calculated.

Let us now consider rock properties typical of limestone; $G_f = 31 \text{ J/m}^2$, $E' = 30 \text{ GPa}$ and $G = 11.25 \text{ GPa}$. To estimate λ , we exploit the similarity of rock to concrete, for which extensive crack shear tests have been conducted. Taking the results of Paulay and Loeber's⁴⁰ tests plotted in Figure 2 of Bažant and Gambarova,⁴¹ we have, for crack opening displacement $\delta_n = 0.125 \text{ mm}$, $\tau/\Delta \approx 40 \text{ N/mm}^2$, which yields for λ the value of 0.25 m . No results seem to be available for $\delta_n = 0$, however, we may use Paulay and Loeber's tests for $\delta_n = 0.25$ and 0.50 mm to approximately extrapolate to 0; this leads to the crude estimate $\lambda = 0.1 \text{ m}$, which we will use. Nevertheless, there is enormous uncertainty about the value of λ , especially for the small initial displacements that matter for initial buckling.

From equation (32), for very large R we have $\sigma_{ef} = 82.7 \text{ MPa}$ (12,000 psi). This means that a sufficiently large borehole would break out at the depth of about 3000 m below the earth surface. This is certainly a reasonable estimate, as an order of magnitude. However, the corresponding value obtained for the spacing of the splitting cracks, which is obtained from equation (30) as 0.25 mm , does not seem reasonable, since at such a small spacing LEFM ceases to be valid and the aforementioned G_f -value, obtained on laboratory samples, is probably inapplicable. But the aforementioned differential equation for buckling degenerates to the form $w'' = 0$, i.e. the slab-columns do not bend at all, which signifies that the idea of buckling makes no sense in the limit case $R \rightarrow \infty$. Probably, the constant C_0 in equation (32) should be interpreted merely as an empirical large-scale compression strength limit, rather than a theoretical value derived by slab buckling analysis.

Next consider a borehole of radius $R = 0.2 \text{ m}$ and assume that $k = 0.25$. Equation (31) then yields $\sigma_{ef} = 21.7 \text{ MPa}$ (3140 psi), which is the stress at the depth of about 740 m. From this result we observe that, if the crack shear resistance were neglected, the predicted breakout stress would be, compared to experience, much too low, by an order of magnitude. This shows that some other mechanism, which we proposed to be the crack shear resistance, must serve to elevate the breakout stress by an order of magnitude. Together with the foregoing value associated with crack shear, equation (33) yields the estimate $\sigma_{ef} = 104.4 \text{ MPa}$ (15,140 psi), which corresponds to depth 3740 m. The thickness and length of the slab-columns are obtained as $h = 2.6 \text{ mm}$ and $L = 50 \text{ mm}$. For such a close spacing, the cracks are more likely to be discontinuous rows of microcracks than continuous cracks, and the crack tortuosity due to heterogeneous microstructure is likely to cause significant local weakening of the slab-columns. In that case, the formula for buckling of a perfect column of a uniform cross-section might be too far from reality and imperfections might have to be introduced into the buckling analysis. Nevertheless, the aspect ratio of the slab columns, $L/h = 19.2$, is certainly just right within the range where the carrying capacity is indeed governed by the theory of buckling of slender columns.

In the preceding numerical estimation, the size-independent part due to crack shear resistance, C_0 , dwarfs the size-dependent part due to bending stiffness, $C_1 h^{-2/5}$. One must be aware, though, of the strong speculative nature of the foregoing estimates. Particularly, the value we used for λ is highly uncertain, and so is the value of k . Consequently, the values of C_0 and C_1 could be quite different, and the magnitude of the size-dependent term could be relatively much more significant than in the foregoing calculation. Experimental studies are needed.

The preceding analysis of crack shear ignored the volume expansion which is always caused by the slip of rough cracks. This expansion is partially prevented by the surrounding rock, which causes hydrostatic compressive stress to develop in the cracking zone. When the volume expansion is not opposed, as in prismatic test specimens with lubricated ends (Appendix I), the crack shear stiffness may be very low, and when it is completely prevented, very high. In addition

to this, the energy of hydrostatic compression needs to be subtracted from the energy that is released from the surrounding rock, which means that less energy is available to create the splitting cracks. This may well be another reason why the collapse stress estimate from the slab-column buckling analysis seems much too low.

6. SUMMARY AND CONCLUSIONS

The basic simplifying hypotheses of the present solution may be summarized as follows:

1. The compression failure of rock on the side of the borehole is caused by densely distributed parallel splitting cracks in the direction of the minimum principal stress, rather than by plastic yielding.
2. The zones of parallel splitting cracks for boreholes of various diameters are geometrically similar and the length of these cracks is proportional to the borehole diameter.
3. For estimating the energy release, the inner boundary of the infinite elastic solid may be considered to expand during failure from a circle to an ellipse.
4. After uniformly spaced splitting cracks parallel to the minimum principal stress develop, the region between the ellipse and the original circle retains a certain residual stress governed by post-critical buckling behaviour of the rock slabs between the cracks.
5. The residual stress value is governed by buckling of rock slabs between the splitting cracks.
6. The buckling stress can be approximately calculated from the average length of the splitting cracks, which is assumed to be proportional to the borehole diameter.
7. Buckling of the slab-columns is resisted not only by their elastic bending stiffness but also by shear stresses produced at the rough crack faces by crack shear.
8. The energy (per unit area) required for crack growth in rock is constant, i.e. independent of the crack length and spacing.

The following basic observations and conclusions can be made:

1. Considering the boundary of the cracking region in borehole breakout to be symmetric and elliptical, and assuming the energy that drives the parallel compression splitting cracks to be released due to buckling of the slabs of rock between the cracks, one can obtain a simple analytical solution for the collapse stress.
2. The dependence of the collapse stress on the spacing of the splitting cracks exhibits a minimum, and the actual crack spacing may be considered to correspond to this minimum.
3. Borehole breakout exhibits a size effect such that, for sufficiently small diameters, the effective breakout stress decreases as the $(-2/5)$ power of the borehole diameter. For sufficiently large diameters, the size dependence disappears.
4. For sufficiently small diameters, the spacing of splitting cracks increases as the $(4/5)$ -power of the borehole diameter, while for sufficiently large diameters a constant spacing is approached.
5. The energy release calculation for a growing ellipse according to Eschelby's theorem also predicts the effect of stress triaxiality, i.e. of the ratio of the remote principal stresses [equation (24)] (which is different from the result obtained by plastic analysis).

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APPENDIX I: COMPRESSION STRENGTH OF A PRISMATIC SPECIMEN

The present use of buckling analysis has been inspired by a similar previous analysis of a prismatic specimen of length L and width b ;³⁰ see Figure 2(b). We assume that compression failure is caused by the formation of a band of vertical splitting cracks of length a and spacing h and is accompanied by buckling of the slabs between the cracks which behave as fixed-end columns. The initial longitudinal stress σ_0 in each slab is reduced by buckling to $\sigma_{cr} = -(E'h^2/3)\pi^2/a^2$. The total energy loss due to buckling is $-\Delta\Pi = Lb(\sigma^2 - \sigma_{cr}^2)/2E'$. The number of cracks is b/h and the energy dissipated by fracture is $\Delta W_f = aG_f b/h$. Energy balance requires that $-\Delta\Pi = \Delta W_f$. From this, the stress required for the formation of the band of splitting cracks is

$$\sigma_0^2 = \frac{2E'G_f a}{h} \frac{1}{L} + \frac{\pi^4 E'^2}{9a^4} h^4 \quad (34)$$

We see that this expression has a minimum as a function of the crack spacing h . From the necessary condition of a minimum, $\partial(\sigma_0^2)/\partial h = 0$, we find that

$$h = \frac{a}{L} \left(\frac{9G_f}{\pi^4 E'} \right)^{1/5} L^{4/5} \quad (35)$$

Substituting this into equation (21), we conclude that the specimen fails at the stress

$$\sigma_0 = C_1 L^{-2/5}, \quad \text{with } C_1 = (\sqrt{27}\pi^2 E'^3 G_f^2)^{1/5} \quad (36)$$

This size effect is the same as found for a borehole. Note also that σ_0 is independent of band width a , which means there is no tendency for the band width to localize.