

Consequences of diffusion theory for shrinkage of concrete*

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The paper reviews the known consequences of diffusion theory for the mathematical modelling of shrinkage, including the size dependence of drying half-time, and the initial and final asymptotic shapes of the time curves of total moisture loss, to which shrinkage is proportional. The mathematical arguments are extended and presented with greater rigour and generality than before, taking the non-linearity of the diffusion equation as well as ageing into account.

1. INTRODUCTION

The evolution of drying shrinkage of concrete depends on the moisture diffusion out of concrete. Diffusion equation solutions were exploited for the description of shrinkage by Carlson [1] and Pickett [2,3], and more recent work established the consequences of diffusion theory for the drying half-time [4,5] and for the initial asymptotic shape of the shrinkage curve in time [6,7], partially taking into account the non-linearity of the diffusion equation due to the dependence of diffusivity on the pore relative humidity. The final asymptotic shape of the shrinkage curve has also been discussed to some extent [6,7]. The purpose of this paper is to summarize the previously deduced consequences, extend the arguments and present them with greater rigour and generality, particularly with regard to the non-linearity of the diffusion equation and the effect of ageing, and derive (under certain simplifying assumptions) the final asymptotic shape of the shrinkage curves. The results will be utilized in a series of papers presenting a shrinkage and creep prediction model, which follows.

2. GENERAL DIFFUSION FORMULATION

The mass conservation equation and the expression for the diffusion flux of moisture, J , are

$$\frac{\partial w}{\partial t} = -\operatorname{div} J \quad J = -C \operatorname{grad} H \quad (1)$$

(for one dimension $\operatorname{div} = \partial/\partial x$ and $\operatorname{grad} = \partial/\partial x$) in which t = time, w = specific water content, H = relative pore vapour pressure (humidity) inside the specimen and C = permeability, which depends on H , i.e. $C = C(H)$, as shown in Fig. 1a. The desorption isotherm is written as $w = g(H)$ = function of H (see Fig. 1b) and from this

$$\frac{\partial H}{\partial t} = k \frac{\partial w}{\partial t} \quad (2)$$

in which $k = 1/g'(H)$ = inverse slope of the desorption isotherm, which also depends on H , i.e. $k = k(H)$.

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Eliminating w and J from Equations 1 and 2, one obtains the non-linear diffusion equation

$$\frac{\partial H}{\partial t} = k \operatorname{div} (C \operatorname{grad} H) \quad (3)$$

The initial condition at the start of drying at age t_0 , and the boundary conditions for a wall of thickness D , are

$$\text{for } t = t_0 \text{ and } 0 \leq x \leq D: \quad H = h_0 \quad (4)$$

$$\text{for } x = 0 \text{ or } D \text{ and } t > t_0: \quad H = h \quad (5)$$

where h is the given constant environmental humidity. From experiments it is known that the average longitudinal shrinkage strain of a sufficiently long specimen is essentially proportional to the average moisture loss per unit volume, that is

$$\varepsilon_{\text{sh}} = \frac{1}{V} \int_V K_s (w - w_0) dV \quad (6)$$

in which V = volume of structure, w_0 = initial specific moisture content before drying and K_s = proportionality coefficient, which is considered as approximately constant although it depends slightly on w or H .

The diffusion problem may be transformed to the non-dimensional variables

$$\xi = \frac{x}{D} \quad \tau = \frac{t - t_0}{\tau_{\text{sh}}} \quad (7)$$

in which

$$\tau_{\text{sh}} = \frac{a_s}{C_1} D^2 \quad (8)$$

Here a_s = constant and C_1 = permeability at saturation, introduced for convenience. Now noting the relations $\partial/\partial t = \tau_{\text{sh}}^{-1} \partial/\partial \tau$, $\partial/\partial x = D^{-1} \partial/\partial \xi$, $\operatorname{grad} = D^{-1} \operatorname{grad}_\xi$ and $\operatorname{div} = D^{-1} \operatorname{div}_\xi$, in which grad_ξ and div_ξ are the gradient and divergence with respect to coordinate ξ , Equations 3 to 5 are transformed to the form

$$\frac{\partial H}{\partial \tau} = \frac{a_s}{C_1} k(H) \operatorname{div}_\xi [C(H) \operatorname{grad}_\xi H] \quad (9)$$

$$\text{for } \tau = 0 \text{ and } 0 \leq \xi \leq 1: \quad H = h_0 \quad (10)$$

$$\text{for } \xi = 0 \text{ or } 1 \text{ and } \tau > 0: \quad H = h \quad (11)$$

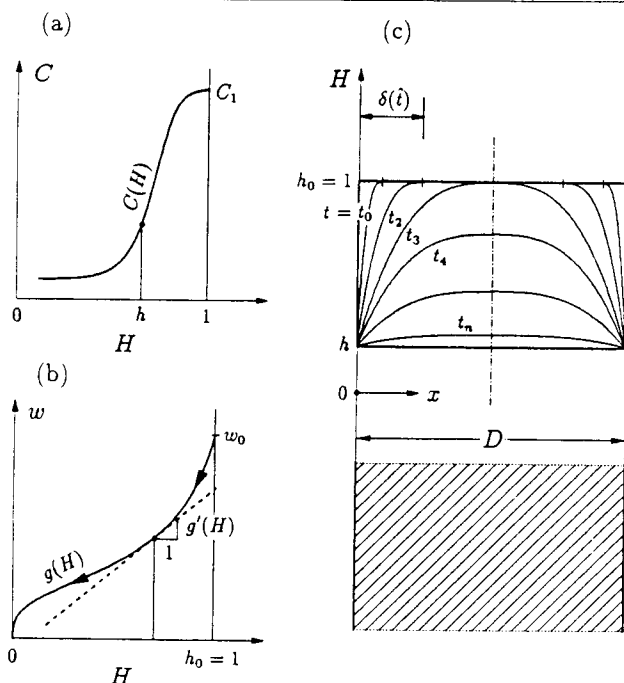


Fig. 1 (a) Dependence of permeability on pore humidity, (b) desorption isotherm, (c) typical humidity distributions in a wall exposed to drying.

The partial differential equation, Equation 9, along with the initial and boundary conditions in Equations 10 and 11, represents an initial boundary-value problem that does not contain D and τ_{sh} ; therefore its solution $H(\xi, \tau)$ is independent of D and τ_{sh} . From Equation 7, $t - t_0 = \tau \tau_{sh}$, that is

$$t - t_0 = \tau \frac{a_s}{C_1} D^2 \quad (\text{for same } H) \quad (12)$$

in which τ is independent of thickness D . This proves that according to the diffusion theory the drying times are proportional to thickness-squared, even if permeability C , or the inverse slope of desorption isotherm k , or both, depend on pore humidity H (or specific water content w). From Equation 12 $\log(t - t_0) = \log(a_s \tau / C_1) + 2 \log D$, which means that the shrinkage curves for two sizes, D_1 and D_2 , are mutually horizontally shifted by distance $2 \log(D_1/D_2)$, if plotted in the logarithmic scale of the drying time.

In the foregoing derivation we ignored the dependence of permeability on age t of concrete. This dependence may be approximately described as

$$C = C_a(t) C_p(H) = t^{-n} C_p(H) = \left(1 + \frac{t}{t_0}\right)^{-n} t_0^{-n} C_p(H) \quad (13)$$

in which C_a and C_p are functions and $n = \text{constant}$. Obviously the age dependence of diffusivity should cause the horizontal shift between two shrinkage curves to be larger than $2 \log(D_1/D_2)$; however, there may be other phenomena that counteract this. For example, the shrinkage stresses in a thicker specimen should be smaller since they are relaxed by creep. This means that a thicker

specimen should suffer less irreversible microcracking than a thinner specimen, and less irreversible microcracking means higher shrinkage, which tends to diminish the distance between the shrinkage curves. Other complicating factors are that ageing in a thicker specimen proceeds faster than in a thinner specimen, because a high humidity persists in it for a longer time. Thus it is not surprising that overall the test data do not seem to indicate any strong systematic deviation from the proportionality of the drying times to D^2 .

Having solved the function $H(\xi, \tau)$, one can integrate the function $w(\xi, \tau)$ from Equation 2. Then one can estimate the overall shrinkage for the case of one-dimensional diffusion in a wall as follows:

$$\varepsilon_{sh} = \frac{2}{D} \int_0^{D/2} K_s(w - w_0) dx = \int_0^{1/2} 2K_s(w - w_0) d\xi \quad (14)$$

The foregoing formulation can be easily adapted to cross-sections that are similar in two or three dimensions. It turns out that the size-squared dependence of the shrinkage times (Equations 8 and 12) remains valid. This is proven by the fact that the derivation of Equations 9 to 11 is valid for the general form of the divergence and gradient operators (since we did not need to introduce their one-dimensional forms). For the cases of a cylinder and a sphere of diameter D , one has

$$\begin{aligned} \varepsilon_{sh} &= \frac{1}{\pi D^2} \int_0^{D/2} K_s(w - w_0) 2\pi x dx \\ &= \int_0^{1/2} 2K_s(w - w_0) \xi d\xi \quad (\text{cylinder}) \quad (15) \end{aligned}$$

$$\begin{aligned} \varepsilon_{sh} &= \frac{3}{4\pi D^3} \int_0^{D/2} K_s(w - w_0) 4\pi x^2 dx \\ &= \int_0^{1/2} 3K_s(w - w_0) \xi^2 d\xi \quad (\text{sphere}) \quad (16) \end{aligned}$$

Denoting as N the number of dimensions (1, 2, 3), we can write Equations 14 and 15 as one equation:

$$\varepsilon_{sh} = k_N \int_0^{1/2} K_s(w - w_0) \xi^{N-1} d\xi \quad (17)$$

where k_N is a constant.

3. CONSEQUENCES FOR INITIAL ASYMPTOTIC SHRINKAGE CURVE

At the beginning of drying, the front of drying, at a distance $\delta(t)$ from the surface, penetrates gradually into the wall. Before it meets the front of drying from the opposite surface, the drying process may be considered the same as in an infinite half-space. Furthermore, whether the surface is straight or curved cannot matter for sufficiently short drying times t because as long as $\delta(t) \ll$ curvature radius R the behaviour must be the same as for a planar surface. Because in a half-space there is no length scale except the depth of penetration of drying, the pore

humidity profiles must be self-similar and have the form

$$H - h_0 = f(\rho) \quad \rho = \frac{x}{\delta(\hat{t})} \quad \hat{t} = t - t_0 \quad (18)$$

where $\delta \ll D/2$, $\delta \ll R$. Because diffusivity C is a function of h and t and not of the drying time $\hat{t} = t - t_0$, C may be considered constant for sufficiently short drying times, $\hat{t} \ll t_0$, as is clear from Equation 13. So we now consider C as constant. According to Equation 18

$$\frac{\partial}{\partial t} = -x \frac{\delta}{\delta^2} \frac{\partial}{\partial \rho} \quad \text{grad } H = \frac{1}{\delta} \text{grad}_\rho H \quad \text{div } H = \frac{1}{\delta} \text{div}_\rho H \quad (19)$$

in which grad_ρ and div_ρ are gradient and divergence with respect to coordinate ρ . Introducing these operators into Equation 3 and simplifying, we obtain the partial differential equation $-\delta \dot{\delta} \rho \partial H / \partial \rho = k \text{div}_\rho (C \text{grad}_\rho H)$, which may be rewritten as

$$\frac{2}{C_1} \delta \dot{\delta} = -\frac{2k}{\rho(\partial H / \partial \rho)} \text{div}_\rho \left(\frac{C}{C_1} \text{grad}_\rho H \right) \quad (20)$$

in which we denote $C_1 = C(1) =$ permeability at saturation, and the superior dot denotes the derivative with respect to time t . Now we notice that the left-hand side of Equation 20 is a function of time only, and the right-hand side is a function of ρ only. This is possible if and only if both sides are equal to a constant, whose value may be denoted as a_0^2 . This achieves separation of variables, and the left-hand and right-hand sides of Equation 20 yield the following two ordinary differential equations:

$$\frac{2}{C_1} \delta \dot{\delta} = a_0^2 \quad (21)$$

$$\rho \frac{\partial H}{\partial \rho} + \frac{2k}{a_0^2} \text{div}_\rho \left(\frac{C}{C_1} \text{grad}_\rho H \right) = 0 \quad (22)$$

The last equation governs the humidity profile in terms of coordinate ρ , and integration of the first equation, with the initial condition $\delta = 0$ for $\hat{t} = 0$, yields for the drying penetration depth the law

$$\delta = a_0 (C_1 \hat{t})^{1/2} \quad (23)$$

According to Equation 14, the overall shrinkage strain of a planar wall may be calculated as follows:

$$\begin{aligned} \varepsilon_{\text{sh}} &= \frac{2}{D} \int_0^{D/2} K_s [w(H) - w_0] dx \\ &= -A_1 \frac{\delta}{D} = -A_0 \frac{\hat{t}^{1/2}}{D} = -A_s \left(\frac{\hat{t}}{\tau_{\text{sh}}} \right)^{1/2} \end{aligned} \quad (24)$$

in which

$$A_1 = - \int_0^1 K_s \{w[h_0 + f(\rho)] - w_0\} d\rho = \text{positive constant}$$

A_0 and $A_s =$ constants, and τ_{sh} is identical to that in Equation 8. This proves that initially the overall shrinkage strain of a wall must evolve as the square root of drying duration, and the shrinkage half-time governing the initial shrinkage must be proportional to thickness

squared. These properties are exact in the asymptotic sense for $\hat{t} \rightarrow 0$. They are not spoiled by considering creep, because for initial shrinkage the creep times are sufficiently short and so creep is negligible. Note also that these properties are valid despite the dependence of diffusivity and the desorption isotherm slope on pore humidity.

A calculation similar to Equation 24 can be made for two-dimensional and three-dimensional similarity. The results are the same, which also ensues from the fact that Equation 17 is valid for any number of dimensions and could have been used in Equation 24 with the same result.

It is interesting that, according to Equation 24, the overall shrinkage strain initially satisfies the differential equation

$$\dot{\varepsilon}_{\text{sh}} = \frac{A_t}{\tau_{\text{sh}} \varepsilon_{\text{sh}}} \quad (\text{for } \hat{t} \ll \tau_{\text{sh}}) \quad (25)$$

where $A_t = A_s^2/2$, i.e. the shrinkage rate declines as an inverse function of the shrinkage strain.

4. CONSEQUENCES FOR FINAL ASYMPTOTIC SHRINKAGE CURVE

At the end of the drying process, the humidity profiles approach a uniform distribution $H = h =$ environmental relative humidity, as shown in Fig. 1c. They may be expressed in the form

$$H - h = \varphi(t) f(\xi) \quad \xi = \frac{x}{D} \quad (26)$$

in which $\varphi(t) =$ function of time, $f(\xi) =$ function of ξ , and $\xi =$ non-dimensional spatial coordinate. Noting that $\hat{t} = t - t_0 = t[1 - (t_0/t)] \simeq t$, it is clear that for long times, i.e. $t \gg t_0$, \hat{t} can be replaced by t , as done in Equation 26. Noting that $\partial/\partial x = D^{-1} \partial/\partial \xi$, i.e. $\text{div} = D^{-1} \text{div}_\xi$, $\text{grad} = D^{-1} \text{grad}_\xi$, where div_ξ and grad_ξ are the divergence and gradient operators with respect to coordinate ξ , we obtain from Equation 3

$$\dot{\varphi} f = k D^{-1} \text{div}_\xi (C D^{-1} \varphi \text{grad}_\xi f) \quad (27)$$

In contrast to the initial asymptotic curve, the final asymptotic curve is affected by the age dependence of diffusivity (Equation 13). However, it is also necessary to consider the fact that, in a thicker wall, a high pore humidity H prevails for a longer time, which causes the ageing (hydration) to proceed faster, which in turn causes the average diffusivity in a thicker wall to increase compared to that in a thinner wall. To account for these effects one may surmise that $k \simeq k_0 t^{-n}$, where n is some empirical constant ($n \geq 0$). Thus Equation 27 becomes

$$t^n D^2 \frac{\dot{\varphi}(t)}{\varphi(t)} = \frac{k_0}{f(\xi)} \text{div}_\xi [C \text{grad}_\xi f(\xi)] \quad (28)$$

in which C is a function of H . Again the left-hand side is a function of time only while the right-hand side is a function of ξ only. This is possible only if both sides are equal to a constant, which for convenience is denoted as

$-\kappa_1 C_1$, where $\kappa_1 = \text{constant}$ and C_1 is the reference initial permeability at saturation. Setting the right-hand side equal to this constant, one gets an ordinary differential equation, whose solution yields the humidity profile $f(\xi)$. Setting the left-hand side equal to $-\kappa_1 C_1$ and integrating, one has

$$\int \frac{d\varphi}{\varphi} = - \int \frac{\kappa_1 C_1}{D^2} t^{-n} dt \quad (29)$$

and evaluating the integrals under the condition that $\varphi \rightarrow 0$ for $t \rightarrow \infty$, one obtains the solution

$$\varphi(t) = \varphi_1 \exp \left[-\kappa_1 \left(\frac{t}{\bar{\tau}_{sh}} \right)^{1-n} \right] \quad (30)$$

in which

$$\bar{\tau}_{sh} = \left(\frac{1-n}{C_1} \right)^{1/(1-n)} D^{2/(1-n)} \quad (31)$$

According to Equation 14, the difference of overall shrinkage ε_{sh} of a wall from the final shrinkage $\varepsilon_{sh\infty}$ may be calculated as

$$\begin{aligned} \varepsilon_{sh} - \varepsilon_{sh\infty} &= K_N \int_0^{1/2} K_s \{ w[h_0 + \varphi(t)f(\xi)] - w_0 \} \xi^{N-1} d\xi \\ &\simeq K_N \int_0^{1/2} K_s w'(h_0) f(\xi) \xi^{N-1} d\xi \varphi(t) \\ &= \varphi(t) \times \text{const} \end{aligned} \quad (32)$$

and then, according to Equation 30,

$$\varepsilon_{sh} - \varepsilon_{sh\infty} = \varepsilon_0 \exp \left[-\kappa_1 \left(\frac{t}{\bar{\tau}_{sh}} \right)^{1-n} \right] \quad (33)$$

This indicates that the approach of shrinkage to its final value ought to be an exponential of a power function of time. From data fitting it appeared that the value $n = 1/2$ gives reasonably good fits; however, since long-time data are very scattered as well as scant, this is merely a weak corroboration. As for the dependence of $\bar{\tau}_{sh}$ on thickness, Equation 31 indicates that the thickness-squared rule can be true only for $n=0$, which differs from the aforementioned n -value. If $n > 0$, one would get $\tau_{sh} \propto D^r$, where $r > 2$, which would mean that the horizontal distance between the shrinkage curves should be increasing with the duration of shrinkage. Although the data show no

clear evidence of that, the value $n=0$ could nevertheless approximately hold at the end of the ageing process assuming that hydration came to a stop. For the sake of simplicity, the presently proposed formulae adhere to the rule $\tau_{sh} \propto D^2$ for the entire shrinkage history.

It is interesting that, according to Equation 33, the final asymptotic phase of shrinkage satisfies the following differential equation:

$$\dot{\varepsilon}_{sh} = \frac{\kappa_0 t^{-n}}{\tau_{sh}} (\varepsilon_{sh\infty} - \varepsilon_{sh}) \quad (\text{for } t \gg \tau_{sh}) \quad (34)$$

5. CONCLUSIONS

The foregoing analysis, which takes into account the non-linearity of the diffusion equation and approximately also ageing while neglecting the effect of microcracking (which will be discussed in the subsequent paper), leads to the following conclusions:

1. The shrinkage (or drying) half-time, τ_{sh} , is proportional to the square of the body size (thickness), D .
2. The initial asymptotic shrinkage time curve is proportional to $(t/\tau_{sh})^{1/2}$, where t is the duration of drying.
3. The final asymptotic shrinkage curve, under certain simplifying assumptions, should approach the final value of shrinkage as a decaying exponential of a power function of time.

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RESUME

Conséquences de la théorie de diffusion pour le retrait du béton

On considère les conséquences de la théorie de diffusion pour la modélisation mathématique du retrait du béton, y

compris l'effet de dimension sur la durée du semi-séchage et la forme asymptotique initiale et finale des courbes de temps de la perte d'humidité totale à laquelle la résistance est proportionnelle. On expose et on présente de façon plus rigoureuse et plus générale qu'auparavant les arguments mathématiques, en tenant compte de la non linéarité de l'équation de diffusion et du vieillissement.