STATISTICAL SIZE EFFECT IN QUASI-BRITTLE STRUCTURES: II. NONLOCAL THEORY

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ABSTRACT: The failure probability of structures must be calculated from the stress field that exists just before failure, rather than the initial elastic field. Accordingly, fracture-mechanics stress solutions are utilized to obtain the failure probabilities. This leads to an amalgamated theory that combines the size effect due to fracture energy release with the effect of random variability of strength having Weibull distribution. For the singular stress field of linear elastic fracture mechanics, the failure-probability integral diverges. Convergent solution, however, can be obtained with the nonlocal-continuum concept. This leads to nonlocal statistical theory of size effect. According to this theory, the asymptotic size-effect law for very small structure sizes agrees with the classical-power law based on Weibull theory. For very large structures, the asymptotic size-effect law coincides with that of linear elastic fracture mechanics of bodies with similar cracks, and the failure probability is dominated by the stress field in the fracture-process zone while the stresses in the rest of the structure are almost irrelevant. The size-effect predictions agree reasonably well with the existing test data. The failure probability can be approximately calculated by applying the failure-probability integral to spatially averaged stresses obtained, according to the nonlocal-continuum concept, from the singular stress field of linear elastic fracture mechanics. More realistic is the use of the stress field obtained by nonlinear finite element analysis according to the nonlocal-damage concept.

INTRODUCTION

In the preceding paper (Bažant et al. 1991), it was shown that the classical Weibull-type analysis of the size effect in brittle failure is invalid for quasi-brittle structures such as reinforced concrete structures, rock masses, ice sheets, or parts made of tough ceramics. This is due to the existence of a large macroscopic stable crack growth prior to reaching the maximum load.

The objective of this paper is to propose a modified statistical theory that takes into account the effect of large macroscopic fractures and distributed cracking at fracture front. This theory, which represents an amalgamation of Weibull theory and the size-effect theory based on energy release, will be shown to agree well with the existing experimental evidence. All the definitions and notations from the preceding paper are retained.

NONLOCAL CONCEPT AND HANDLING OF STRESS SINGULARITY

As argued in the preceding paper (Bažant et al. 1991), the stress-distribution function to be used in the integral for failure probability of the structure must be the stress distribution at incipient failure, rather than some stress distribution that exists long before failure. This distribution must reflect the localization of strains and stresses that occur prior to reaching

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the maximum load. In the extreme case, a complete localization of cracking, a sharp crack develops upon reaching the maximum load, as illustrated in Fig. 1. The stress distribution is then singular and has the form:

$$\sigma_i = \sigma_N \rho^{-1/2} \phi_i(\rho, \theta) \quad \text{(1)}$$

in which $\sigma_i(i = 1, \ldots, n)$ = the principal stresses; $\rho = r/D$; D = characteristic dimension of the structure; ρ, θ = polar coordinates centered at the tip of the crack; $\sigma_N = F/bD$, where b = structure thickness; F = applied load; σ_N = the nominal strength (nominal stress at maximum load); $\phi_i(\rho, \theta)$ = a nonsingular function that is bounded, continuous, and smooth (except at concentrated loads and at boundary corners).

Stress singularity is an abstraction that does not exist in reality. The stresses near the tip of a sharp crack are blunted due to inelastic phenomena such as microcracking or other damage, which are the consequence of heterogeneity of the material. One effective way to take this heterogeneity into account is the nonlocal-continuum concept (Kröner 1967; Eringen 1965, 1966, 1972), which is properly applied only to the variables associated with failure or damage (Bažant and Pijaudier-Cabot 1988; Bažant and Lin 1988a, 1988b; Bažant and Ožbolt 1990). Thus, recognizing that failure at a point of a heterogeneous material must depend not only on the continuum stress at that point but also on the stress resultant or average stress within a certain representative volume V_R of the material, we realize that the probability of failure should not depend on the local stresses $\sigma_i(\mathbf{x})$ but on the average stresses

$$\bar{\sigma}_i(\mathbf{x}) = \int_{V_R} \sigma_i(\mathbf{s}) W(\mathbf{x} - \mathbf{s})\, dV(\mathbf{s}) \quad \text{(2)}$$

in which $W(\mathbf{x} - \mathbf{s})$ = a given empirical weight function, which must satisfy the normalizing condition, $\int_{V_R} W(\mathbf{x} - \mathbf{s})\, dV = 1$. When the representative volume V_R protrudes across the body boundary or the crack boundary, the protruding part must be chopped off and the weights α must be scaled up,

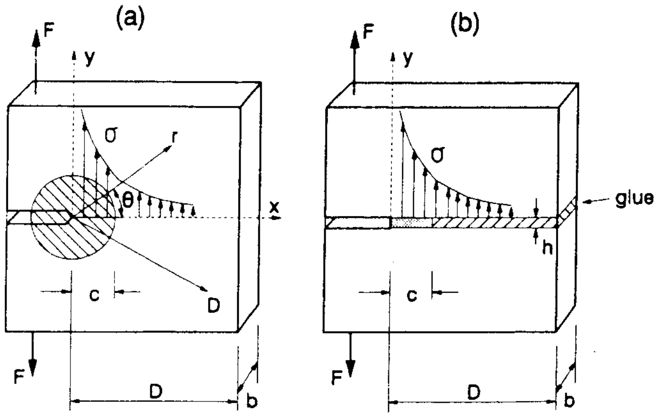

FIG. 1. Stress Distributions and Crack Process Zone in Specimens

so as to satisfy the normalizing condition. The weight function introduces the nonlocal material properties. For the special case that W = Dirac δ-function, one has $\bar{\sigma}_i(\mathbf{x}) = \sigma_i(\mathbf{s})$, which is the case of local continuum. For zero strength threshold ($\sigma_u = 0$), an equation from the companion paper (Bažant et al. 1991) may now be rewritten as

$$-\ln(1 - P_f) = \int_V \sum_{i=1}^{n} \left(\frac{\bar{\sigma}_i(\mathbf{x})}{\sigma_0}\right)^m \frac{dV}{V_r} \quad \text{(3)}$$

There is also another argument for the nonlocal concept—the fact that the random material properties in adjacent small material elements cannot be uncorrelated. The spatial integral in (4) does not introduce spatial correlation of failure probabilities for any two points for which the domains of integration in (4) overlap, i.e., stresses $\bar{\sigma}_i(\mathbf{x}^A)$ and $\bar{\sigma}_i(\mathbf{x}^B)$ such that their averaging integrals both involve the same stress $\sigma_i(\mathbf{x}^C)$.

Eqs. (2) and (3) are too difficult for an analytical solution. Therefore, one must take the nonlocal aspect into account in the simplest possible manner. As is known from the previous studies, the nonlocal averaging (2) is unnecessary in the regions of small damage, i.e., far from the fracture front (large ρ). However, within volume V_c of the fracture-process zone (shaded in Fig. 1), some form of nonlocal averaging is necessary. As an approximation, we may consider a constant average stress value $\bar{\sigma}_i$ through this entire zone, which is physically justified by the stress limit posed by inelastic deformation (and is similar to a yield limit). Eq. (3) may thus be simplified as follows

$$-\ln(1 - P_f) = \sum_{i=1}^{n} \left(\frac{\bar{\sigma}_i}{\sigma_0}\right)^m \frac{V_c}{V_r} + \int_{V_e} \sum_{i=1}^{n} \left(\frac{\sigma_i}{\sigma_0}\right)^m \frac{dV}{V_r} \quad \text{(4)}$$

in which $V_e = V - V_c$ = volume of the rest of the body outside the fracture-process zone.

One may now be tempted to consider the value of $\bar{\sigma}_i$ as a constant yield stress. However, regardless of the absence of a clearly defined yield limit in the material, this would be incorrect since the $\bar{\sigma}_i$-value must be considered a random variable and must be determined also on the basis of extreme value distribution such as Weibull's, in relation to the random nominal strength σ_N. Therefore, it is proposed to approximate $\bar{\sigma}_i$ as the value of the elastically calculated stress σ_i at a point that lies on the crack extension line ($\theta = 0$) at a certain fixed distance $r = c$ from the tip of the ideal sharp crack. Thus, for the region $V_c(\rho \le c/D)$, we introduce the approximation

$$\bar{\sigma}_i \simeq k(\sigma_i)_{\substack{r=c\\ \theta=0}} = \phi_N \left(\frac{c}{D}\right)^{-1/2} k\phi_{ci}, \qquad \phi_{ci} = \phi_i\left(\frac{c}{D}, 0\right) \quad \text{(5)}$$

in which D = characteristic dimension (size of the structure); k and c = empirical constants; and c may be interpreted as the effective radius of the fracture-process zone [see Bažant and Kazemi (1990)], which in turn is related to the characteristic length of the nonlocal-continuum model approximating the heterogeneous material (Bažant and Pijaudier-Cabot 1988, 1989). The value of $\bar{\sigma}_i$ is random because σ_N is random.

Approximate Nonlocal Analysis

In the integral (4), we have $dV = bD^2\rho\, d\theta\, d\rho$, where b = thickness of the body. This expression applies for the case of two-dimensional similarity.

To be more general and cover also the case of three-dimensional similarity, one may generalize the foregoing expression as follows:

$$dV = b_0 D^n \rho \, d\theta \, d\rho \quad \text{. . . . . (6)}$$

in which b_0 = nondimensional constant; and n = number of spatial dimensions. For two-dimensional similarity ($n = 2$), we have $b = b_0$, and for three-dimensional similarity ($n = 3$), $b = b_0 D$. In the case of axisymmetric fracture situations, b_0 is a constant without any meaning of thickness. The effect of the number of dimensions on the first expression in (4) is a more difficult question. Strictly speaking, the volume of the fracture-process zone is $V_c = \pi c^2 b_0$ ($b_0 = b$) for two dimensions ($n = 2$) and $V_c = \pi c^2 b_0 D$ for three dimensions ($n = 3$), which in general may be written as $V_c = \pi c^2 b_0 D^{n-2}$. For three dimensions, however, this would mean that the cracks would propagate independently in various parts of the fracture-process zone throughout the thickness of the fracture specimen, which is impossible except if the fracture-process zone is very thick ($b >> c$). It seems more reasonable to assume that once the crack forms or propagates, it must do so simultaneously throughout the whole thickness b of the specimen, so that the probability of survival depends on the fracture-process zone area rather than the volume. Consequently, we will assume that, for the purpose of survival probability in (4),

$$V_c = \pi c^2 b_0 D^p \quad \text{. . . . . (7)}$$

where $p = 0$. Later, however, we will also explore the case $p = n - 2$.

Substituting (5) and (7), we may rearrange (4) as follows

$$-\ln(1 - P_f) = \left[A_0 \left(\frac{D}{c}\right)^{(p + m/2)} + A_1 \left(\frac{D}{c}\right)^n H_D \, I\left(\frac{c}{D}\right) \right] \sigma_N^m \quad \text{. . . . . (8)}$$

in which $A_0 = \pi b_0 c^{p+2} k^m \sigma_0^{-m} \Sigma_i \varphi_{ci}^m / V_r$, $A_1 = b_0 c^n \sigma_0^{-m} / V_r$, which are size-independent constants; H_D = parameter to be defined later, and

$$I\left(\frac{c}{D}\right) = \int_{-\pi}^{\pi} \int_{c/D}^{\hat{\rho}(\theta)} \sum_{i=1}^{n} [\rho^{-1/2} \phi_i(\rho,\theta)]^m \rho \, d\rho \, d\theta, \qquad \text{except if } c << D \quad \text{. . (9)}$$

Here function $\hat{\rho}(\theta)$ represents the relative radial distances to the boundary points and defines the geometry of the structure. For geometrically similar specimens, the value of $I(c/D)$is constant.

Furthermore, it has been necessary to introduce in (8) parameter H_D which takes into account interaction of the fracture process zone with the specimen boundaries. For the extreme cases of very large and very small specimens relative to the fracture-process zone size c, one must obviously have

$$\text{For } \frac{c}{D} << \hat{\rho}(\theta): \qquad H_D = 1 \quad \text{. . . . . (10}a\text{)}$$

$$\text{For } \frac{c}{D} \approx \hat{\rho}(\theta): \qquad H_D = 0 \quad \text{. . . . . (10}b\text{)}$$

Eq. (10*a*) pertains to the case where the fracture-process zone size is negligible compared to the structure size (dimension) D. Eq. (10*b*) pertains to the case where the fracture-process zone extends roughly over the entire structure (or at least the entire ligament length). In that case the domain V_e of the integral in (4) is vanishing, and the vanishing of the integral is ensured by setting $H_D = 0$. For the intermediate body sizes, for which the fracture-process zone is neither small nor large, parameter H_D must be considered to be a smooth function of the relative size of the process zone, c/D, i.e., $H_D = H_D(c/D)$. We leave it as an empirical function, although it could conceivably be calculated after solving the boundary value problem in its nonlocal form [(2) and (3)].

From (9), one can easily demonstrate that the singular stress distribution of linear elastic fracture mechanics cannot be used in the integral for failure probability, i.e., in (8) of the preceding paper (no doubt this is the reason why the stress redistributions due to large cracks have so far been ignored in the literature on Weibull-type size effect). For c/D approaching zero, the integral in (9) converges if and only if $m \leq 4$. But this is not a realistic situation. The typical value of Weibull modulus for concrete is $m \approx 12$ (with $\sigma_u = 0$). Therefore, it is inevitable to consider c finite, as proposed in this paper.

The preceding consideration shows that the stress distribution in (2), with σ_i proportional to σ_N, is unrealistic for the fracture-process zone when the body is very large, i.e., $D >> c$. For that case the stresses at the ideal crack tip become very large and would exceed the intrinsic strength f^* of the material with no flaws or microcracks. Therefore, the stress must be limited as $\sigma_i \leq f^*$ and must be taken as $\sigma_i = f^*$ when (2) exceeds this value. From this consideration it follows that, for $c << D$, the integral in (4) should be proportional to f^{*m}, i.e., a constant. This may be achieved by setting

$$\text{For } c << D: \qquad I\left(\frac{c}{D}\right) \approx k_0 \left(\frac{f^*}{\sigma_N}\right)^m \quad \text{. . . . . (11)}$$

in which k_0 = some constant.

If the failure probability P_f is fixed, (8) yields for the size effect the following relation

$$\sigma_N = \left\{ \frac{1}{-\ln(1 - P_f)} \left[A_0 \left(\frac{D}{c}\right)^{(p + m2)} + H_D A_1 I(c/D) \left(\frac{D}{c}\right)^n \right] \right\}^{-1/m} \quad \text{. . . (12)}$$

For $P_f = 0.5$, (12) yields the median strength, which is known from experiments on concrete to be close to the mean strength.

Let us now discuss the limiting cases, assuming that $p = 0$ and $m/2 > n$ ($n = 2$ or 3). For very small D/c, $(D/c)^{m/2} << (D/c)^n$, while for very large D/c, $(D/c^{m/2} >> (D/c)^n$. Also note that $I(c/D)$ tends to a finite value for very small sizes (11) because σ_i is bounded. So the first term in the bracket of (12) must dominate for very large sizes, while the second term must dominate for very small sizes. Thus we obtain the following asymptotic size effect laws (with $p = 0$):

$$\text{For small } D/c: \qquad \sigma_N \propto D^{-n/m} \quad \text{. . . . . (13)}$$

$$\text{For large } D/c: \qquad \sigma_N \propto D^{-1/2} \quad \text{. . . . . (14)}$$

In the foregoing asymptotic results, it is particularly noteworthy that for very large structures (14) the size effect is the same as that obtained by deterministic analysis on the basis of energy release, i.e., $\sigma_N \propto D^{-1/2}$. The reason is that, for large structures, the stress peak in the fracture-process zone dwarfs the stresses in the rest of the structure, thus making the statistical variability of the strength in the rest of the structure irrelevant. Only the statistical variability of strength within the fracture-process zone matters. But the size of this zone is essentially fixed, independent of the body dimensions. Consequently, the statistical part of the size effect disappears and the only size effect left is deterministic—namely that due to the increase in the energy release rate as the structure size (along with the fracture length at imminent failure) is increased.

Eq. (14) for large D/c is based on the hypothesis that in the fracture-process zone a crack must form or propagate simultaneously within the entire specimen thickness b. For very thick specimens, however, it is conceivable that a crack might form or propagate only in a part of thickness and be stationary in the remaining part of thickness. In that case, as already explained, it might be more appropriate to use some value of p between 0 and $p = n - 2 = 1$ in (7) for three-dimensional similarity ($n = 3$). Eq. (14) would then be replaced by:

For large c/D: $\sigma_N \propto D^{-(p/m + 1/2)}$ (three-dimensional similarity only) . . (15)

Assuming $m = 12$, we would have $0 < p/m < 1/12$, which is relatively small correction. But it is noteworthy that a statistical size effect slightly stronger than
$D^{-1/2}$ cannot be ruled out. Experimentally, though, it has not been observed in tests with three-dimensional similarity (torsion, pull-out, punching shear), and so we will not consider this possibility any further.

To calculate the mean strength $\bar{\sigma}_N = E(\sigma_N)$, in which E = expectation, one may use the following well-known relation

$$\bar{\sigma}_N = E(\sigma_N) = \int_0^1 \sigma_N \, dP_f = \int_0^\infty (1 - P_f) d\sigma_N \quad (16)$$

which follows from the fact that the cross-hatched area in Fig. 2 can be obtained by integrating either over the horizontal strips or over the vertical strips. Now, denoting the expression in the square brackets in (12) as Z, one obtains [from (12)] $1 - P_f = \exp(-Z\sigma_N^m)$. So (16) yields:

$$\bar{\sigma}_N = \int_0^\infty \exp(-Z\sigma_N^m) \, d\sigma_N = Z^{-1/m} \frac{1}{m} \int_0^\infty t^{1/m - 1} e^{-t} dt$$

$$= Z^{-1/m} \Gamma\left(1 + \frac{1}{m}\right) \quad (17)$$

in which Γ = Gamma function. Substituting the bracketed expression in (12) for Z, one gets for the mean strength the following size effect law

$$\bar{\sigma}_N = \Gamma\left(1 + \frac{1}{m}\right) \left[A_0 \left(\frac{D}{c}\right)^{m/2} + H_D A_1 I\left(\frac{c}{D}\right) \left(\frac{D}{c}\right)^{n} \right]^{-1/m} \quad (18)$$

As we see, except for a factor, this law is formally identical to the size-effect law in (11) for a fixed failure probability P_f. The limiting behavior for very small and very large sizes is, for the mean strength, again the same as that in (13) and (14). By a similar procedure, one can easily also calculate the variance of σ_N.

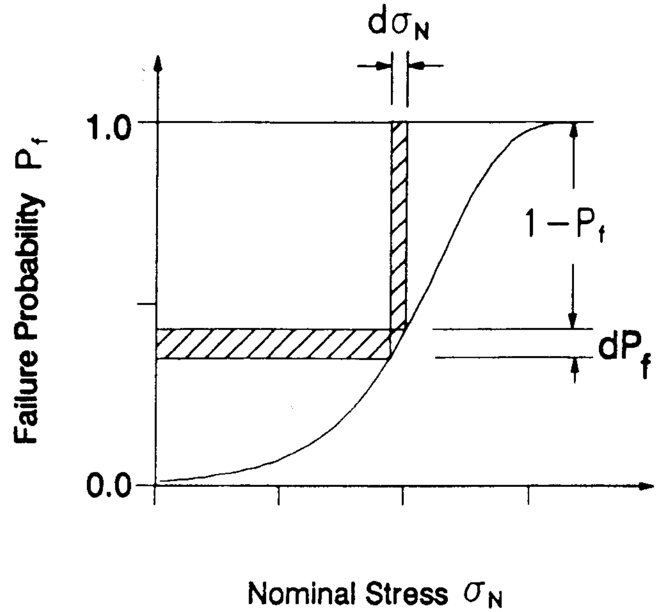

FIG. 2. Cumulative Distribution of Failure Probability of Structure

SPECIAL CASE OF ONE-DIMENSIONAL SIMILARITY

In the preceding paper (Bažant et al. 1991) we saw that one-dimensional similitude, exemplified by failure of a glue layer between two blocks, gives a different law of size effect. Consider now a similar case but with a stress singularity produced by an existing crack or notch—the specimen in Fig. 1(b), consisting of two strong parts glued by a thin layer, the strength of which is much less than the strength of the adjacent parts. Let the thickness b be the same for specimens of various sizes D. Similar to (2), the normal stress in the glue layer is $\sigma = \sigma_N \rho^{-1/2} \varphi(\rho)$ where $\rho = x/D$ [Fig. 1(b)] and φ is a continuous smooth function. Similar to (5), the nonlocal stress in the fracture-process zone is approximated as $\bar{\sigma} = k(\sigma)_{r=c} = \sigma_N (c/D)^{-1/2} k \varphi(c/D)$. Substituting this into (3) and integrating over the volume of the thin glue layer, we get

$$-\ln(1 - P_f) = \left[A_0 \left(\frac{D}{c}\right)^{m/2} + A_1 H_D I\left(\frac{c}{D}\right) \frac{D}{c} \right] \sigma_N^m \quad (19)$$

in which $A_0 = bhc\sigma_0^{-m} k^m \phi^m (c/D)/V_0$, $A_1 = bhc\sigma_0^{-m}/V_0$ and

$$I\left(\frac{c}{D}\right) = \int_{c/D}^1 \rho^{-m/2} \phi^m(\rho) \, d\rho \quad (20)$$

and coefficient H_D (10) is introduced for the same reasons as before. By the same arguments as those that lead from (12) to (13) and (14), we may conclude that the term with A_1 dominates for very large sizes and the term with A_2 dominates for very small sizes. Therefore, For small D/c: $\sigma_N \propto D^{-n/m}$ .... (21)

For large D/c: $\sigma_N \propto D^{-1/2}$ .... (22)

This agrees with (13) and (14), since $n = 1$ in this case.

Empirical Interpolation between Asymptotic Size Effects

The asymptotic size-effect laws for very small and very large sizes in (13) and (14) are very simple. More importantly, they are also independent of the geometry of the structure. The complete size-effect law described by (12) is of little value because the intermediate variation of coefficients I and H_D between the asymptotic cases is difficult to determine, and is sure to depend on the geometry of the structure. Therefore, (12) cannot be proposed for practical use.

From extensive experimentation (Bažant and Kazemi 1989; Bažant and Sener 1988; Bažant et al. 1990; Bažant and Cao 1987; Gettu et al. 1990) it appears that, as a good approximation relative to the scatter of the test result, the form of the size-effect law can be considered as shape-independent through the entire range of sizes up to about 1:20. This has been confirmed by the success of the approximate size-effect law

$$\sigma_N = \frac{Bf_t'}{\sqrt{1 + \beta}}, \qquad \beta = \frac{D}{D_0} \quad \text{.............................. (23)}$$

proposed in Bažant (1984), with Bf_t' and D_0 as empirical constants (f_t' is the direct tensile strength, introduced to make B nondimensional). In view of this observation, it is not inappropriate to obtain the complete statistical size-effect law for bodies with large fractures by a simple empirical interpolation formula that agrees with both asymptotic cases in (13) and (14) and at the same time reduces to (23) for $m\ \infty$ (the deterministic limit). The simple formula with these asymptotic properties is

$$\sigma_N = \frac{Bf_t'}{\sqrt{\beta^{2n/m} + \beta}} \quad \text{.............................. (24)}$$

Possibly one might consider a more general interpolation formula:

$$\sigma_N = \frac{Bf_t'}{(\beta^{2nr/m} + \beta^r)^{1/2r}} \quad \text{.............................. (25)}$$

where r = an arbitrary empirical constant, analagous to that considered in Bažant (1987) and Bažant and Pfeiffer (1987) for deterministic size effect. But for the deterministic case, Bažant and Pfeiffer found $r = 1$.

Determination of Material Parameters

Same as for (23), it is possible to identify parameters D_0 and B by linear regression of σ_N-data for geometrically similar specimens of various sizes. To this end, we may algebraically rearrange (25) to the linear plot $Y = AX + C$ in which

$$X = D^{r-(2rn/m)}, \qquad Y = \left(\frac{f_t'}{\sigma_N}\right)^{2r} D^{-2rn/m},$$

$$A = B^{-2r}D_0^{-r}, \qquad C = D_0^{-2rn/m}B^{-2r} \quad \text{.............................. (26)}$$

(n = the given number of dimensions). Thus, if r and m are known, A and C, along with the coefficients of variation of A and of the deviations of the data points from the regression line, can be determined from the plot of Y versus X by linear regression, and B and D_0 follow from (26). As for the value of r, one needs to try various values and then determine that for which the coefficient of variation of the deviations from the regression line is minimum. The linear regression previously introduced for the deterministic size-effect law (23) is the special case of (26) for $m \to \infty$, $r = 1$.

Similar to the previous use of (23) (Bažant and Pfeiffer 1987; Bažant et al. 1989), the regression results for B can be used to determine the fracture energy, G_f, defined as the critical energy release rate in an infinitely large fracture specimen. According to LEFM, one has $G_f = F^2g(\alpha)/E'b^2D$ where $E' = E/(1 - \nu^2)$ for plane strain, $E' = E$ = Young's modulus for plane stress (ν = Poisson ratio); α = relative crack length; and $g(\alpha)$ = nondimensional energy release rate for the given specimen geometry, which can be easily determined by elastic analysis. Inserting $F = \sigma_N bD$, one gets $\sigma_N = [E'G_f/g(\alpha)d]^{1/2}$. From (24) or (25), the asymptotic behavior for large D is $\sigma_N = Bf_t'/\sqrt{\beta}$, while $\alpha \to \alpha_0$ (initial crack or notch length). Setting these two expressions equal, one gets

$$G_f = \frac{B^2 f_t'^2 d_0}{E'} g(\alpha_0) \quad \text{.............................. (27)}$$

which happens to coincide with the expression obtained previously from (23). This means that the statistical strength effects on G_f are nil, which is not surprising.

Similar to Bažant and Kazemi (1990), one could also deduce from (24) or (25) the effective length c_f of the fracture-process zone length, but this value is affected by the Weibull statistical parameters m and σ_0.

Question of Weibull Modulus *m* for Fracture-Process Zone

In our derivation, Weibull's distribution of material strength has been used in a somewhat different sense than in the classical problem of failure of a long fiber or chain. Is our value of Weibull modulus m the same as for direct tension tests of long fibers or bars? Probably not, and probably it is larger. The reason is that Weibull modulus m increases with increasing uniformity of the distribution of the flaws in the material (Freudenthal 1968), and the distribution of flaws that must be considered is that just before failure. The uniformity of flaws in the fracture-process zone should be much higher than in the initial state of the material. Therefore, the value of m may be larger than the value obtained from direct tension tests of bars of different lengths, which is about 12. However, due to the absence of test results that would suffice for determining m, we will consider in the numerical examples that follow the value $m = 12$ because it is conservative. The larger the m-value, the weaker is the size effect. In future research, however, an estimate of the m-value for the fracture-process zone should be obtained.

Comparison with Test Results

To be able to check the asymptotic trend defined by (13) and (14) and the empirical interpolation formula in (24), test data of a rather broad size range are needed. Although no known data are perfect in this regard, the recent data in Fig. 3(a) are of a sufficiently broad range. They show the optimum fits by (23) (nonstatistical) and (24) (statistical) for the test results of Bažant and Kazemi (1989) on diagonal shear failure of longitudinally reinforced concrete beams without stirrups. The size range of these data, 1:16, was quite broad. We see that the statistical (24) fits these data somewhat better than the deterministic (23). The difference between these two size-effect laws is discernible only for small specimen sizes, and the small-size asymptotic slope $-n/m$ seems to be acceptable.

In evaluating these tests, there is a question with regard to the failure mode of the smallest specimen, which was of a different type: it is essentially a bending-type failure, while all the large beams exhibited typical diagonal shear failure [see Bažant and Kazemi (1989)]. In the present phenomenologic analysis of failure, this would hardly be a sufficient argument for excluding the smallest beam data. So it seems justified to keep the smallest beam data in Fig. 5(a). But if one speaks strictly of diagonal shear, these data must of course be excluded. For that case, the optimum fit by (23) and (24) is shown in Fig. 3(b), which is however not too different from Fig. 3(a).

To check how good the interpolation formula in (24) and (25) might be, one may for example use the exact solution for the stress distribution in an infinite space with a crack of length $2a$, subjected to remote uniaxial stress at infinity [Fig. 4(a)]. The solution (Westergaard 1939) is

$$\sigma_{1,2} = \frac{\sigma}{\sqrt{|R|}}\left(x \cos \frac{\theta + 2k\pi}{2} - y \sin \frac{\theta + 2k\pi}{2}\right) \pm \frac{y\sigma a^2}{(|R|)^{3/2}} \quad \text{(28)}$$

where

$$R^2 = (x^2 - a^2 - y^2)^2 + 4x^2y^2 \quad \text{(29)}$$

$$\theta = \bar{\theta} \text{ if } x^2 - a^2 - y^2 > 0$$

$$\theta = -(\pi - \bar{\theta}) \text{ if } x^2 - a^2 - y^2 < 0 \quad \text{(30)}$$

$$\bar{\theta} = \arctan \frac{-2xy}{x^2 - a^2 - y^2} \quad \text{(31)}$$

We now consider square specimens with centric cracks of length $2a$. To obtain the simple stress field in (28), we assume that the rectangular boundary is subjected to traction distributions that correspond to the stress field according to (28). (The stress at infinity, σ, no longer exists but represents merely the nominal stress σ_N, playing the role of a parameter of the loading by these boundary tractions.) The stress field in (28) [Fig. 5(a)] has been subjected to the spatial averaging according to (2), using the weight function $W(\mathbf{x}) = [1 - (r/R)^2]^2$ for $r < R$; $W = 0$ for $r \geq R$, in which $R = \sqrt[3]{3/4}\, l = 0.9086\, l$, and l = characteristic length of nonlocal continuum [a material property whose measurement was reported by Bažant and Pijaudier-Cabot (1989)]. The averaging has been carried out by representing the solution from (28) by nodal values of a square mesh and approximating the integral in (2) by a discrete sum. This has been done for various sizes of the rectangular specimen, using of course the same value of l (and R) for each size. The results of nonlocal averaging is plotted in Fig. 5(b). The optimum fits by (23) and (24) are shown in Fig. 4(b). It is clear that (24) gives the

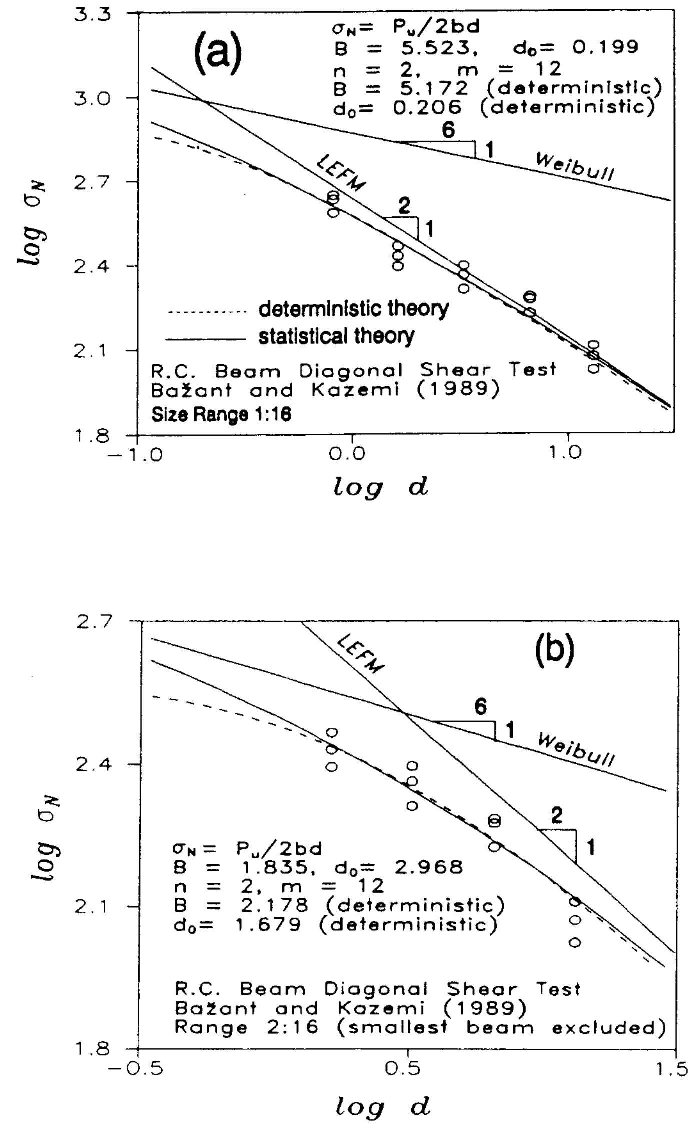

FIG. 3. Test Result on Diagonal Shear Failure of Reinforced Concrete Beam without Stirrups, and Optimum Fits by (23) and (24) (*a*) Is for Five Beams; (*b*) Is for Four Beams

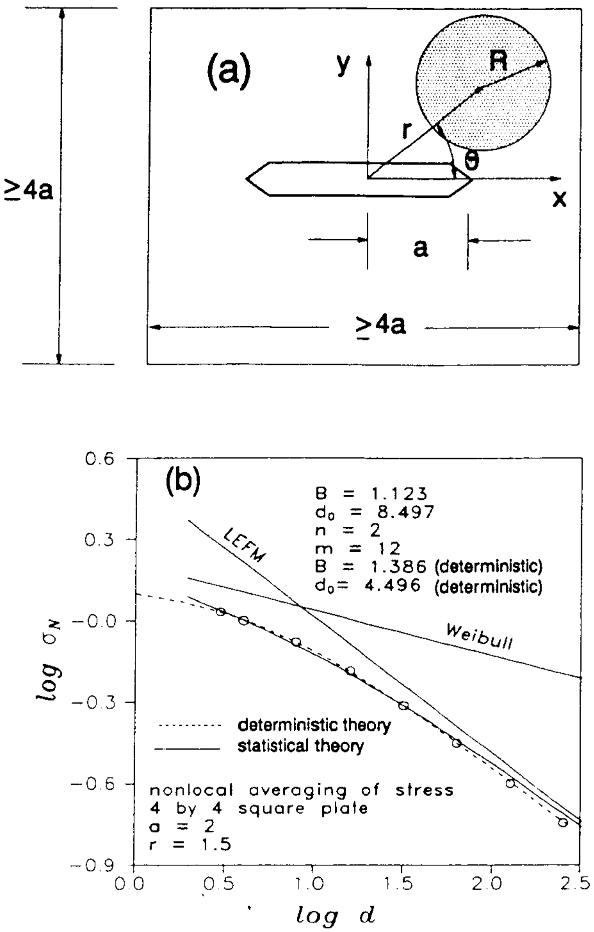

FIG. 4. Numerical Result by Nonlocal Averaging and Linear Finite Element Analysis on Square Specimen with Centric Cracks (*a*), and Optimum Fit by (23) and (24) (*b*). Singular Stress Field (c) and Stress Field After Nonlocal Averaging (*d*)

better fits of the test results and that the shape of these curves is quite satisfactory.

Failure Probability Based on Nonlocal Damage Analysis by Finite Elements

A more realistic, but also more complex, approach is that of nonlocal damage. In that case, the failure probability at a point of the body may be assumed to be governed by the spatially averaged (nonlocal) principal strains

$$\bar{\varepsilon}_i(\mathbf{x}) = \int_V \varepsilon_i(\mathbf{x}) W'(\mathbf{x},\mathbf{s}) dV(\mathbf{s}) \quad \text{(32)}$$

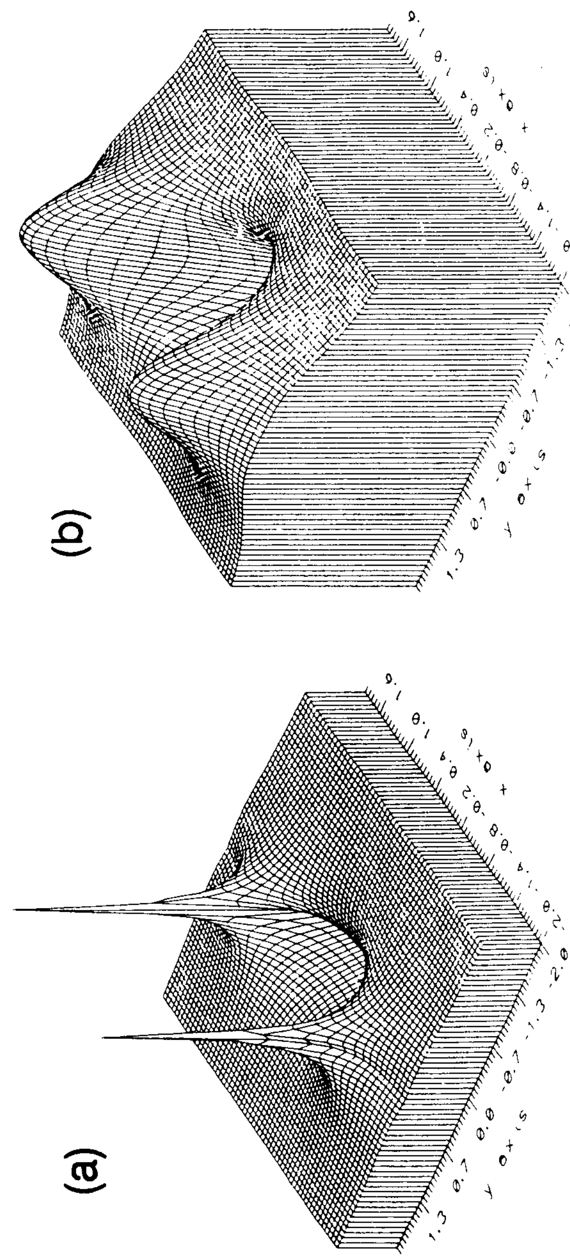

FIG. 5. Numerical Result by Nonlinear Nonlocal Damage Model on Rectangular Specimen, and Optimum Fit by (23) and (24)

where ε_i = principal strains (local); $W'(\mathbf{x},s)$ = given nonlocal weighting function based on the characteristic length of the material (Bažant and Pijaudier-Cabot 1988). The averaging can be carried out with uniform weight (W' = constant) over a circle, as shown in Fig. 3(*a*) (however, near the crack surface or boundary, the protruding part must be chopped off). But it seems more appropriate to consider W' as a smooth bell-shaped function, declining smoothly to zero with the distance from the point (see Bažant and Ožbolt 1990). The values of $\bar{\varepsilon}_i(\mathbf{x})$ may be obtained from a nonlinear nonlocal finite element program as shown (Bažant and Lin 1988a, 1988b). It is important to note that in this approach, the failure probability must be considered to be a function of strain rather than stress. The reason is that the nonlocal damage admits strain softening, i.e., a decrease of stress with increasing strain, so that the failure probability considered as a function of stress would decrease rather than increase during the post-peak strain-softening deformation. In elastic analysis of fracture, dependence on stress or strain is of course equivalent.

The nonlocal damage concept automatically yields a finite size of the fracture-process zone, a basic ingredient of the present theory. I of this study is, in this approach, replaced by:

$$1 - P_f = \exp\left[-\int_V \sum_{i=1}^{n} \left\langle \frac{k_n E \bar{\varepsilon}_i(\mathbf{x}) - \sigma_u}{\sigma_0} \right\rangle^m \frac{dV(\mathbf{x})}{V_r}\right] \quad \text{(33)}$$

where E = the initial Young's elastic modulus; k_n = empirical constant, and $\bar{\varepsilon}_i(\mathbf{x})$ = principal strain field from finite element program for nonlocal damage. The results of these calculations are shown in Fig. 4, and one can see that the result is good.

Limitations of Proposed Theory

The survival probability of the structure is the joint probability of survival of all its elementary parts. The basic hypothesis of the classical Weibull-type theory is that the probability of survival of each part depends only on the stress in that part and is independent of the stresses as well as of the loading history in all the other structure parts. None of this, of course, can be expected to be exactly true. The survival probabilities of various elementary parts are not, in reality, independent. For example, the survival probability of one elementary part may depend also on the stress in the adjacent parts. This dependence is approximately described by the nonlocal concept [which leads to (3)], and by the approximate form of (5) and (8). Moreover, the failure is not a sudden, random event. Rather, it is a random process, and a fully realistic theory would have to consider the probabilistic nature of the steps in this process; for example, the question how the survival probability of one elementary part is influenced by the preceding failure of an adjacent elementary part. In particular, this requires following the incremental jumps of the fracture process in a probabilistic manner.

In the present theory, the failure probability is calculated from an instantaneous picture of the strain field at one critical moment of the loading process. Probability, however, does not enter the calculation of the process that leads to this critical moment. This is certainly a simplification. For a fully realistic method of probabilistic analysis, one would have to consider the probabilities in each loading step. When the major crack extends to a certain point, one would have to consider the probabilities of various propagation directions, and the probabilities of the lengths of the jump of the crack tip for a given load increment. Weibull-type probabilistic considerations have been made for the crack jump process [e.g., Bruckner and Munz (1984) and Chudnovsky and Kunin (1987)]. Numerical simulations have been made as well. But it seems hardly possible to obtain in this manner the general trends such as (13) and (14).

Conclusions

1. The Weibull-type integral that yields the failure probability must be based on the stress field at imminent failure, which reflects the stress redistributions due to large fracture. However, the singular stress field obtained according to linear elastic fracture mechanics cannot be used, since the probability integral diverges.
2. Divergence of the probability integral is avoided in general by the nonlocal concept, in which the failure probability at any point depends not only on the stress at that point but also on the stresses in a small neighborhood of the point.
3. Nonlocal statistical analysis of failure probability indicates that the classical Weibull-type size-effect law (applicable only to bodies without large fractures) is valid asymptotically for very small specimen sizes, while the size-effect law of linear elastic fracture mechanics is approached asymptotically for very large sizes. The failure probability for a very large size of a structure with a macroscopic crack proportional to the structure size is dominated by the stress field in the fracture-process zone and is nearly independent of the stress field in the rest of the structure.
4. The aforementioned asymptotic behavior as well as the presently proposed empirical interpolation formula (24) agrees reasonably well with the existing test data. But the agreement cannot be regarded as a proof.
5. Estimates of the failure probability can be obtained by using the probability integral spatially averaged stress values determined according to the nonlocal-continuum concept from the singular stress field calculated according to linear elastic fracture mechanics. Still more realistic is the use of the stress field obtained by nonlocal nonlinear finite element analysis.
6. Finally, the reasons that, for very large specimens, the present statistical theory yields a nonstatistical size effect, the same as linear elastic fracture mechanics (conclusion 3), may be summed up as follows: (1) The mechanics of localization causes the failure to depend in the limit of infinite size solely on the properties of the fracture-process zone; (2) the size and state of this zone, in an infinitely large specimen, becomes independent of the structure size (as well as shape).

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Appendix I. Structures That Can Fail by One of Many Large Cracks

Our analysis has been predicated upon the hypothesis that, at imminent failure, only one major crack has the potential of growing. Conceivably, it might happen that, at imminent failure, there are many large cracks, each of which is in a critical state and has the potential of growing, and that the crack that grows is decided by chance. For the sake of simplicity, we now restrict attention to very large structures, for which only the stresses in the fracture-process zone very near the tips of large fractures are decisive. In the presence of many large cracks ($k = 1,2,...N_c$), (1) and (5) for the nonlocal averaged value of the principal stress σ_i in the fracture-process zone of the kth crack may be approximately expressed as

$$\bar{\sigma}_{ik} = \sigma_N\left(\frac{c}{D}\right)^{-1/2} k\phi_{ik}\left(\frac{c}{D}, 0\right) \quad \text{(34)}$$

in which ϕ_{ik} = nonsingular smooth functions describing the stress state near the tip of each crack. Substituting this into (3) and assuming, for the purpose of integration, that the stress value given by (28) governs for each entire fracture-process zone, we get

$$-\ln(1 - P_f) \simeq \frac{V_c}{V_r}\sum_{k=1}^{N_c}\left[\frac{\sigma_N}{\sigma_0}\left(\frac{c}{D}\right)^{-\frac{1}{2}} k\phi_{ik}\left(\frac{c}{D}, 0\right)\right]^m \quad \text{(35)}$$

Instead of the summation over all the crack tips, we can introduce the average function $\bar{\phi}_i$ of all the functions ϕ_{ik}. We also make the assumption that the number of large cracks that can potentially cause failure is proportional to the volume of the structure, i.e.,

$$N_c \simeq N_0 D^n \quad \text{(36)}$$

in which n = 2 for two-dimensional similarity and n = 3 for three-dimensional similarity. Then (35) may be rewritten as

$$-\ln(1 - P_f) \simeq \frac{V_c}{V_r} N_0 D^n\left[\frac{\sigma_N}{\sigma_0}\left(\frac{c}{D}\right)^{-\frac{1}{2}} k\bar{\phi}_i\left(\frac{c}{D}, 0\right)\right]^m \quad \text{(37)}$$

Solving this for the nominal stress at failure, we get for very large structure sizes the following asymptotic size-effect law

$$\sigma_N = B_0 D^{-(1/2 + n/m)} \quad \text{(38)}$$

in which B_0 = a constant,

$$B_0 = \frac{-\ln(1 - P_f)\sigma_0^m V_r c^{m/2}}{V_c N_0 k\bar{\phi}_i^m\left(\frac{c}{D}, 0\right)} \quad \text{(39)}$$

The size effect in (38) is stronger than that according to linear elastic fracture mechanics, which is the asymptotic case for the previous formulation. The reason for this difference is that now we assume that each of many large cracks can cause failure, with the number of large cracks being proportional to the structure volume. By contrast, our previous analysis implies that even if there are many large cracks, the mechanics of the problem dictates that only one of them grow during failure. In view of the latest researches on stable path of interactive system and associated thermodynamic stability criterion (i.e., Bažant 1988a, 1988b; Bažant and Cedolin 1991) it is generally found that even if there are two or more cracks in a critical state, only one of them can grow as a stable response path. This is a manifestation of the tendency to localization of failure—a general characteristic of the damage and fracture processes.

For the case of very small structures, the size effect again must asymptotically conform to the classical Weibull-type theory (13). The intermediate size effect must represent a continuous transition from (13) to (38). A simple formula for this transition, which is a special case of that proposed in Bažant (1987), is as follows

$$\sigma_N = Bf_t' \frac{\beta^{-n/m}}{\sqrt{1 + \beta}}, \qquad \beta = \frac{D}{D_0} \quad \text{(40)}$$

At present, however, it is questionable whether any situations exist where this function might be appropriate. None of the existing test data show the final asymptotic slope of the curve of $\log(\sigma_N)$ versus $\log(D)$ to be any steeper than $-1/2$.

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