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SMEARED-TIP SUPERPOSITION METHOD FOR NONLINEAR AND TIME-DEPENDENT FRACTURE

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ABSTRACT: A crack with bridging stresses is treated as a superposition of many cracks whose tips are continuously distributed (smeared) along the crack line. The solution is reduced to an integral equation for the components of the applied load associated with crack tips at various locations. This equation, which is equivalent to that previously presented by Planas and Elices (1986), is then generalized to include: (1) time-dependent nonlinear stress-displacement relation for the bridging stresses (rate-effect), and (2) aging viscoelastic behavior of the material in the rest of the structure. The solution leads to an integro-differential equation, whose method of solution by finite differences in space and time is given. The paper presents only the mathematical formulation. Numerical studies applied to concrete, rock and ceramics are planned.

INTRODUCTION

Although the finite element and boundary element methods are powerful and general approaches to fracture problems with nonlinear and time-dependent material properties, the usefulness of analytical or semianalytical approaches cannot be denied. These methods offer much deeper insight into the mechanics of fracture and are more efficient, more accurate, foolproof and unlikely to fail. For this reason, Green's function methods or solutions based on dislocations are widely used. These methods, however, become too difficult for complex geometries as well as complex material laws.

This paper will present a new semianalytical method based on superposition of exact solutions of linear elastic fracture mechanics. This method, which will treat the crack as a superposition of many linearly elastic cracks whose tips are continuously distributed (smeared) along the crack line, will represent an extension of an integral equation presented by Planas and Elices (1986) to arbitrary nonlinear and time-dependent (or rate-dependent) material behavior in the crack bridging zone, coupled with aging linearly viscoelastic behavior in the rest of the structure. While Planas and Elices used asymptotic series expansions to solve the aforementioned integral equation, we will discuss solving this equation and more general integro-differential equations by finite differences, after properly taking care of stress singularities.

LINEAR ELASTIC FRACTURE MECHANICS RELATIONS FOR A SINGLE CRACK

The stress intensity factor K for a crack of length s in an elastic structure may generally be expressed in the form (e.g. Broek, 1986)

$$K = P\ k(\eta)\ /\ b\ \sqrt{L} \tag{1}$$

where P = applied load, L = length of the crack ligament, b = structure thickness, and $k(\eta)$ = nondimensional function of $\eta = s/L$ where s = coordinate of the crack tip (Fig. 1). We will consider only mode I (opening mode) fracture, although the present method could equally well be applied to mode II or III (shear) fracture. The load-point displacement u may generally be calculated as (e.g. Bažant and Cedolin, 1990) $u = C_0P + P\psi(\eta)/E'b$ or

$$u = P\ \chi(\eta)\ /\ E'b \tag{2}$$

with

$$\chi(\eta) = \overline{C}_0 + \psi(\eta)\ , \qquad \psi(\eta) = 2\int_0^{\alpha} [\ k(\eta')\ ]^2\ d\eta' \tag{3}$$

where $\overline{C}_0 = C_0E'b$, C_0 = compliance when there is no crack; $\chi(\eta)$ is a nondimensional function (which can also be obtained by LEFM, for example by optimum fitting of finite element results); and E' = E = Young's modulus, for the case of plane stress, or $E' = E/(1-\nu)^2$, for the case of plane strain. For a crack with the tip at s, the linear elastic fracture mechanics (LEFM) solutions for the normal stress σ across the crack plane and the crack opening displacement v have the general form:

$$\text{for } \eta < \xi \le 1: \qquad \sigma = P\ S(\xi,\eta)\ /\ Lb \tag{4}$$

$$\text{for } 0 \le \xi \le \eta: \qquad v = P\ V(\xi,\eta)\ /\ E'b \tag{5}$$

where the following approximations can be used for sufficiently small $|\xi-\eta|$:

$$S(\xi,\eta) = k(\eta)\ /\ \sqrt{2\pi(\xi - \eta)} \tag{6}$$

$$V(\xi,\eta) = k(\eta)\ \sqrt{8/\pi}\ \sqrt{\eta - \xi} \tag{7}$$

(Fig.1). When $|\xi-\eta|$ is not small, one has (for $\xi<1$) the asymptotic expansions:

$$S(\xi,\eta) = [b_0 + b_1(\eta)(\xi-\eta) + b_2(\eta)(\xi-\eta)^2 + \ldots]\ k(\eta)\ /\ \sqrt{\xi - \eta} \tag{8}$$

$$V(\xi,\eta) = [c_0 + c_1(\eta)(\eta-\xi) + c_2(\eta)(\eta-\xi)^2 + \ldots]\ k(\eta)\ \sqrt{\eta - \xi} \tag{9}$$

where $b_0 = 1/\sqrt{2\pi}$, $c_0 = \sqrt{8/\pi}$; $b_1(\eta)$, $c_1(\eta)$, $b_2(\eta)$,...are nondimensional smooth continuous functions which can be determined from LEFM, for example by optimum fitting of finite element results with Eqs. 8 and 9.

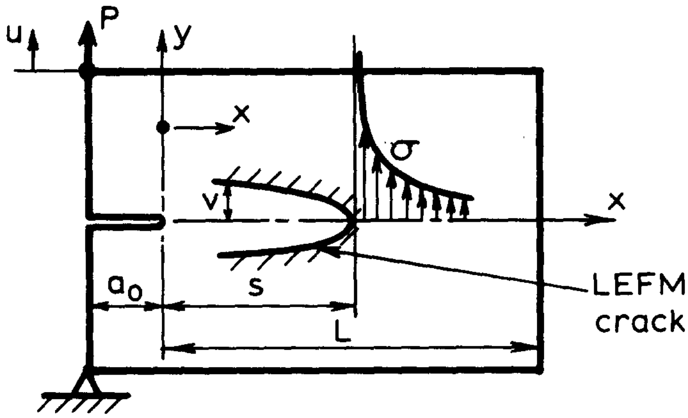

Fig. 1 Stresses and displacements caused by a crack with tip at x = s.

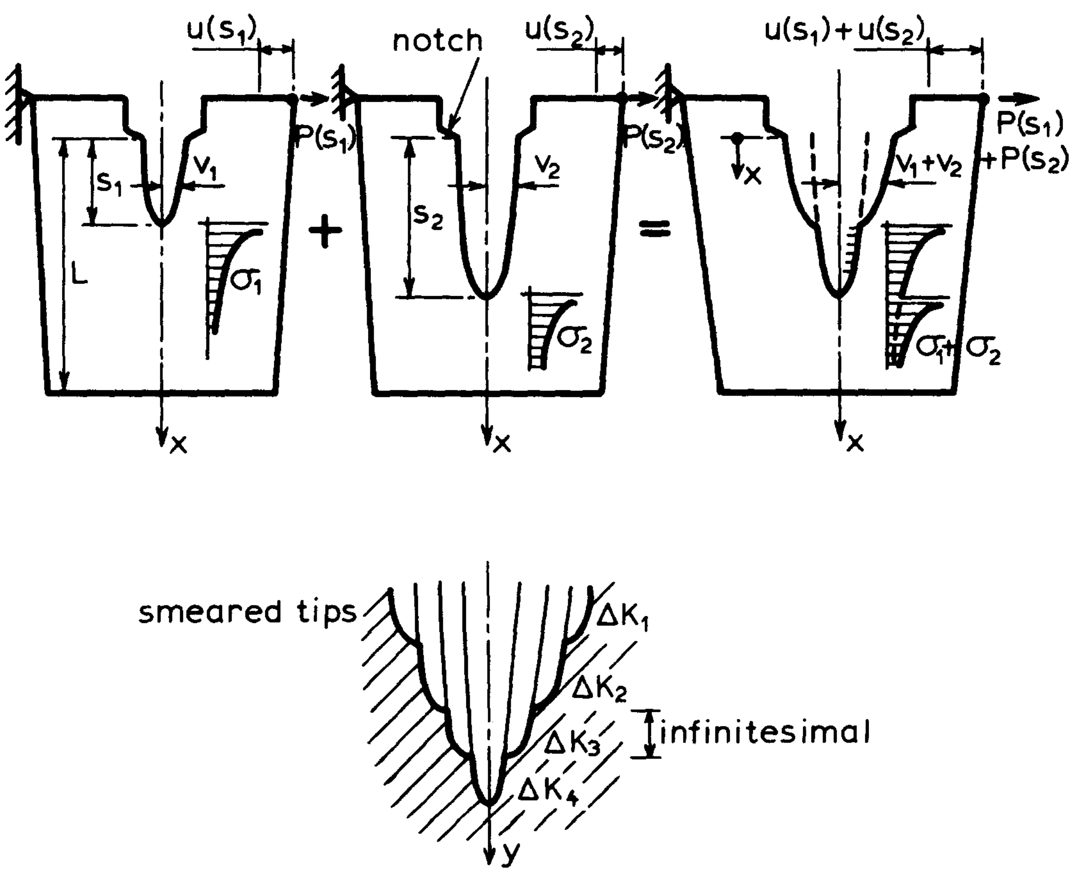

Fig. 2 Superposition of stress and displacement fields of several cracks of different lengths.

SUPERPOSITION OF THE FIELDS OF SMEARED CRACK TIPS

Now we turn attention to nonlinear fracture mechanics of a line crack with a nonlinear fracture process zone of nonzero length. However, the solid outside the crack line is still assumed to be elastic, and nonlinearity arises only from the relation between the crack bridging stress σ and the crack opening displacements v (the width of opened crack is 2v). The opening profile v(x) of such a crack can be obtained as a sum of the opening profiles of many LEFM cracks of stress intensity factors $dK(s) = k(\eta)dP(s)/b\sqrt{L}$ where $\eta=s/L$; the tips of these cracks, of coordinates s, are continuously distributed (smeared) along the crack line (x is the crack length coordinate); see Fig. 2. The stress fields of all these LEFM cracks as well as the corresponding loads must also be superposed.

Let $P(s) = p(\eta)bL$ be the load corresponding to one LEFM crack whose crack tip has the coordinate $s = \eta L$; $p(\eta)$ represents the density of the loads with respect to the relative crack tip coordinate $\eta = s/L$. The load corresponding to the crack tips located within an infinitesimal segment ds with a center at coordinate s is $dP = bLp(\eta)d\eta$ (keep in mind that all the load components dP are applied at the same point as the actual applied load P). By superposition,

$$\sigma(\xi) = \int_0^{\xi} S(\xi,\eta)\ p(\eta)\ d\eta \tag{10}$$

$$v(\xi) = (L/E')\int_{\xi}^{1} V(\xi,\eta)\ p(\eta)\ d\eta \tag{11}$$

To satisfy equilibrium, the sum of all the load components dP corresponding to all the segments ds must be equal to the applied load P, i.e.

$$P = b\ L\int_0^1 p(\eta)\ d\eta \tag{12}$$

The load-point displacement corresponding to the crack tips located within segment ds centered at s is $du = \chi(\eta)p(\eta)L/E'$. Superposition of all these displacement contributions (which all occur at the load application point) yields

$$u = (L/E')\int_0^1 \chi(\eta)\ p(\eta)\ d\eta \tag{13}$$

Eqs. 10–12 are equivalent to those presented by Planas and Elices (1986, 1987; see Eqs. 11–13 in their 1986 paper).

GENERALIZATION TO ARBITRARY NONLINEAR TIME-DEPENDENT FRACTURE LAW

We introduce the nondimensional bridging stresses and opening displacements:

$$\bar{\sigma} = \sigma\ /\ f'_t\ , \qquad \bar{v} = 2v\ f'_t\ /\ G_f \tag{14}$$

where f'_t = tensile strength of the material, and G_f = fracture energy of the material ($G_f = K_{If}^2\ /\ E'$ where K_{If}= fracture toughness); Fig. 3. We will consider a rather general stress-displacement law for the fracture process

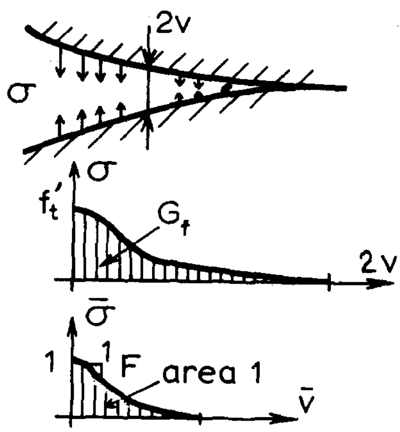

Fig. 3 Crack bridging stresses and stress-displacement relation.

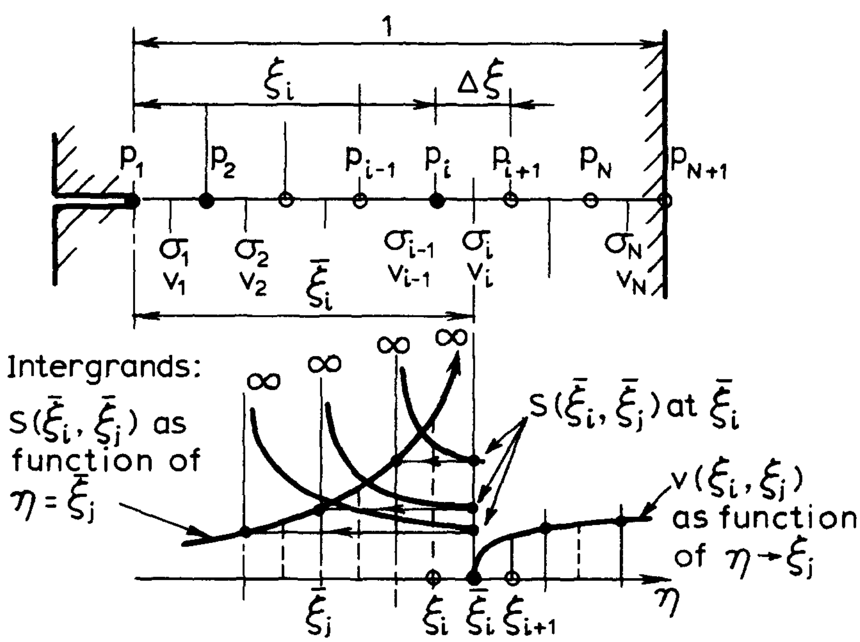

Fig. 4 Discrete subdivision of crack ligament.

zone written in the form:

$$\dot{\bar{v}} = F(\bar{\sigma},\bar{v})\dot{\bar{\sigma}} + \Phi(\bar{\sigma},\bar{v}) \tag{15}$$

where the superior dots denote partial derivatives with respect to time t, and F, Φ are given nondimensional functions characterizing the material. Function Φ describes the rate effect on fracture, and function F the instantaneous nonlinear response. Eq. 15 applies only to virgin loading of the crack. For unloading or reloading we replace $F(\bar{\sigma},\bar{v})$ by 0, i.e.

$$\dot{\bar{v}} = \Phi\ (\bar{\sigma},\bar{v}) \tag{16}$$

Before stress σ for the first time reaches the strength limit f'_t,

$$\dot{\bar{v}} = 0 \tag{17}$$

Substitution of Eqs. 10-11 into Eq. 15 now yields

$$(2Lf'_t/E'G_f) \int_\xi^1 V(\xi,\eta)\ \dot{p}(\eta,t)\ d\eta = F[\bar{\sigma}(\xi,t),\ \bar{v}(\xi,t)]\ f_t'^{-1}\int_0^\xi S(\xi,\eta)\ \dot{p}(\eta,t)\ d\eta + \Phi[\bar{\sigma}(\xi,t),\ \bar{v}(\xi,t)] \tag{18}$$

This represents a singular integro-differential equation. If $\Phi > 0$ and if $\sigma(\xi)$ and $v(\xi)$ are known, then the unknown function $\dot{p}(\xi,t)$ may be solved from this equation. But $\sigma(\xi)$ and $v(\xi)$ are unknown, and so Eqs. 18, 13, 10 and 11 are coupled. If $\Phi = 0$ while $\sigma(\xi)$ and $v(\xi)$ are known and $\dot{u}(t)$ is prescribed, function $\dot{p}(\xi,t)$ may be solved from Eqs. 18 and 13; and if P(t) is prescribed, then from Eqs. 18 and 12.

METHOD OF SOLUTION BY FINITE DIFFERENCES

Time t is subdivided by discrete times t_r (r=1,2,...) into small steps $\Delta t = t_r - t_{r-1}$. The spatial coordinate ξ is subdivided by ξ_i (i=1,2,...N+1) into N small intervals $\Delta\xi = 1/N = \xi_{i+1} - \xi_i$ (Fig. 4). The second-order finite difference approximation of Eq. 18 at interval center point $\bar{\xi}_i = \xi_i + \Delta\xi/2$ and midstep $t = \bar{t}_r = t_r - \Delta t/2$, representing the expression for $\Delta t\ \dot{v}(\bar{\xi}_i,t)$, is:

$$\Delta\xi \frac{2L}{\ell_0} \sum_{j=i+1}^{N} V(\bar{\xi}_i,\bar{\xi}_j)\ \Delta\hat{p}_j + \frac{2L}{\ell_0}\sqrt{8/\pi}\ k(\xi_{i+1})\ \frac{2}{3}\left(\frac{\Delta\xi}{2}\right)^{3/2} \Delta p_{i+1} = F^* \left[\Delta\xi \sum_{j=1}^{i-1} S(\bar{\xi}_i,\bar{\xi}_j)\ \Delta\hat{p}_j + \frac{k(\bar{\xi}_i)}{\sqrt{2\pi}}\sqrt{2\ \Delta\xi}\ \Delta\hat{p}_i \right] \qquad (i=1,\dots N) \tag{19}$$

where $F^* = F$, $\bar{\xi}_i = \xi_i + \Delta\xi/2$, $\Delta\hat{p}_i = (\Delta p_i + \Delta p_{i+1})/2$ and $\ell_0 = E'G_f/f_t'^2$. Also

$$\frac{L}{E'} \sum_{j=1}^{N+1} c_j\ \chi(\xi_j)\ \Delta p_j = \Delta u \tag{20}$$

c_i are the coefficients of the numerical integration formula chosen to evaluate the integral in Eq. 13, and subscripts i, j refer to the discrete coordinates. In deriving Eq.19, the integral in Eq. 18 in the near-tip intervals has been evaluated as follows:

$$\int_{\xi_i}^{\bar{\xi}_i} (\bar{\xi}_i - \eta)^{-1/2} d\eta = \sqrt{2\Delta\xi}\ , \quad \int_{\bar{\xi}_i}^{\xi_{i+1}} (\eta - \bar{\xi}_i)^{1/2} d\eta = \frac{2}{3}\left(\frac{\Delta\xi}{2}\right)^{3/2}$$

If Δu is given, Eqs. 19 and 20 represent a system of N+1 equations for N+1 unknowns Δp_1, Δp_2, $\ldots\Delta p_{N+1}$. After solving the equations, the load increment according of Eq. 12 is

$$\Delta P = b\, L\, \Delta\xi \sum_{j=1}^{N} c_j\, \Delta p_j \tag{21}$$

Note that if the finite difference approximations were centered at ξ_i rather than at interval center points, there would be N+1 equations for N unknowns, and the equation system would be insoluble.

Because function F depends on the unknowns, iterations in each time step must be used. In the first iteration, F* is taken as the value of F calculated from the final values of σ and v from the preceding time step. In the subsequent iterations, F* is taken as the value of F calculated from the midstep values of σ and v obtained in the preceding iteration.

The system of N+1 equations in Eq. 19 may be concisely rewritten as

$$\frac{2L}{\ell_0} \sum_{j=1}^{N+1} \alpha_{ij}\, \Delta p_j = F^* \sum_{j=1}^{N+1} \beta_{ij}\, \Delta p_j \qquad (i=1,2,\ldots N) \tag{22}$$

where α_{ij} and β_{ij} are coefficient matrices which must be generated on the basis of Eq. 19, and are independent of Δp_j and of F*.

GENERALIZATION TO AGING VISCOELASTIC MATERIAL

For materials such as concrete, there is not only a rate effect in the fracture process zone, but also significant creep in the entire structure. Assuming the stresses outside the fracture process zone to be in the linear viscoelastic range, we may consider the stress-strain relation for concrete to be of the form:

$$\underset{\sim}{\varepsilon}(t) = \underset{\sim}{B} \int_0^t J(t,t')\, d\underset{\sim}{\sigma}(t') \tag{23}$$

where $\underset{\sim}{\varepsilon}$ and $\underset{\sim}{\sigma}$ are 6x1 column matrices of stress and strain components, $\underset{\sim}{B}$ is a constant 6x6 matrix implementing the conditions of isotropy, and $J(t,t')$ is the compliance function representing the strain at age t caused by a uniaxial stress applied at age t'. Considering time step $\Delta t_r = t_r - t_{r-1}$ and approximating the history integral in Eq. 23 by a finite sum, one can obtain the quasielastic incremental stress-strain relation:

approximating the history integral in Eq. 23 by a finite sum, one can obtain the quasielastic incremental stress-strain relation:

$$\Delta \underset{\sim}{\varepsilon}_r = \underset{\sim}{B} \, [\, (\Delta \underset{\sim}{\sigma}_r / E'') + \Delta \underset{\sim}{\varepsilon}''_r \,] \tag{24}$$

where

$$\Delta \underset{\sim}{\varepsilon}''_r = \frac{\underset{\sim}{B}}{E''} \sum_{q=1}^{r-1} C_{r,q} \, \Delta \underset{\sim}{\sigma}_q \tag{25}$$

Here $C_{r,q}$ are coefficients which can be deduced from the finite sum approximation to the integral in Eq. 23 (see Eq. 2.33, p.116, Bažant, Ed., 1988). It may now be noted that the elastic stress-strain relation $\underset{\sim}{\varepsilon} = \underset{\sim}{B} \, \underset{\sim}{\sigma} / E$ may be transformed into Eq. 24 by replacing 1/E' with the matrix difference operator:

$$\underline{E}^{-1}(\ldots)_r = \frac{1}{E'} (\ldots)_r + \sum_{q=1}^{r-1} C_{r,q} (\ldots)_r \tag{26}$$

Making now this replacement in Eq. 22, we obtain the following equation

$$\sum_{j=1}^{N+1} \left(\frac{\alpha_{ij}}{E''} + \frac{F^*}{L} \beta_{ij} \right) \Delta p_{j,r} = H_i \qquad (i=1,2,\ldots N) \tag{27}$$

where

$$H_i = - \sum_{j=1}^{N+1} \alpha_{ij} \, \Delta \omega_{j,r} \tag{28}$$

and

$$\Delta \omega_{j,r} = \sum_{q=1}^{r-1} C_{r,q} \, \Delta p_{j,q} \tag{29}$$

Together with Eq. 20, Eq. 27 represents a system of N+1 linear algebraic equations for N+1 unknowns $\Delta p_{j,r}$ (j=1,...N+1). Solving in each time step this equation system, one can obtain, step by step, the load history from the prescribed values of the load-point displacement increments. Alternatively, if the load increments are prescribed, the system of equations for each time step must be enlarged by Eq. 20.

GENERALIZATION FOR MULTIAXIAL BEHAVIOR IN FRACTURE PROCESS ZONE

The preceding formulation tacitly presumed uniaxial stress-displacement relation for the fracture process zone. Since this zone is treated as a line crack with bridging stresses, multiaxial stresses in fact do not exist in this zone. In reality, however, the fracture process zone has some finite width, in which case it is conceivable that, for example, the normal stress σ_x in the direction parallel to the crack may influence the response. Such behavior might produce volume dilatancy in the fracture process zone, which could for example be the driving force of axial splitting cracks produced by uniaxial compression. The present formulation can be also extended to model such behavior if functions F and Φ (Eq. 13) are made to depend also on σ_x or ε_x.

CONCLUSION

The formulation presented makes it possible to obtain accurate solution to fracture problems with rate-dependence of fracture and viscoelasticity of the material, coupled with nonlinear behavior due to bridging stresses in the fracture process zone (e.g. Mazars and Bažant, eds., 1988). However, this formulation still remains to be tested and verified by numerical experience. This will be the next phase of research.

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REFERENCES

Bažant, Z.P., and Cedolin, L. (1990), "Stability of Structures: Elastic, Inelastic and Fracture Theories", Oxford University Press, New York (in press).

Broek, D. (1986), "Elementary Engineering Fracture Mechanics", Martinus Nijhoff, Dordrecht - Boston.

Bažant, Z.P., Ed. (1988), "Mathematical Modeling of Creep and Shrinkage of Concrete", J. Wiley & Sons, Chichester - New York.

Mazars, J., and Bažant, Z.P., Eds. (1988), "Cracking and Damage", Elsevier Applied Science, London - New York 1989.

Planas, J., and Elices, M. (1986), "Un nuovo metodo de analysis del comportamiento asintotico de la propagacion de una fisura cohesiva en modo I" Anales de Mecánica de la Fractura (Madrid), No. 3, pp. 219-227.

Planas, J., and Elices, M. (1987), "Asymptotic analysis of the development of a cohesive crack zone in mode I loading for arbitrary softening curves", Internal Report, Dept. of Physics of Materials, Universidad Politecnica de Madrid; summary presented orally at SEM-RILEM Intern. Conf. on Fracture of Concrete and Rock in Houston, 1987.