

# STABLE STATES AND PATHS OF STRUCTURES WITH PLASTICITY OR DAMAGE

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**ABSTRACT:** Stability conditions for irreversible structural systems are formulated on the basis of the second law of thermodynamics. It is found that distinction must be made between stable equilibrium states and stable equilibrium paths. An equilibrium state is stable if any admissible deviation from it leads to a decrease of the internal entropy of the structure. Among all the equilibrium paths emanating from a bifurcation point, the stable path is that which maximizes the increment of internal entropy. These criteria are expressed in terms of the second-order incremental work, and distinction between load control and displacement control is made. Shanley's perfect elastoplastic column is analyzed as an example. It is found that the undeflected states of the column are stable up to the reduced modulus load. However, the undeflected stable states above the tangent modulus load are not reachable by a continuous loading process. The stable path is such that the deflection becomes nonzero as soon as the tangent modulus load is exceeded. Uniaxial strain-softening must localize right after the peak stress state even though the limit of stable homogeneous strain states occurs only later at a finite descending slope. The results indicate a need to include checks for path stability in inelastic finite element programs, especially those for damage with strain-softening.

## INTRODUCTION

Although the conditions of uniqueness and bifurcation of the equilibrium path of an elastoplastic structure have been studied extensively (Bruhns and Mielniczuk 1977; Bruhns 1984; Hill 1959, 1961, 1962; Hutchinson 1974; Mróz 1966; Shanley 1947; Tvergaard 1985), no basic principles and methods seem to exist for stability of equilibrium states and paths, except for a few special solutions (e.g., Bažant 1976, 1977, 1985, 1986). The principle of minimum potential energy, unfortunately, does not apply to inelastic structures. As we will see, static stability analysis can nevertheless be accomplished on the basis of the second law of thermodynamics. We will show it first for the classical problem of buckling of an elastoplastic column. Based on a report by Bažant (1987), we will then formulate general criteria of stable state and stable path, and finally we will consider some broader implications, especially for softening structures.

## SHANLEY'S ELASTOPLASTIC COLUMN

For the sake of simplicity, let us study the idealized perfect column considered in the epoch-making paper by Shanley (1947); see Fig. 1(b-d). The column is hinged and geometrically perfect. It consists of two rigid bars of lengths  $l/2$ , which are connected by a very short elastoplastic link (point hinge) of length  $h \ll l$  and width  $h$ , having an ideal I-beam cross section of area  $A$ . The lateral deflection and the axial displacement at the load point (positive if shortening) are denoted as  $q_1$  and  $q_2$ , respectively.

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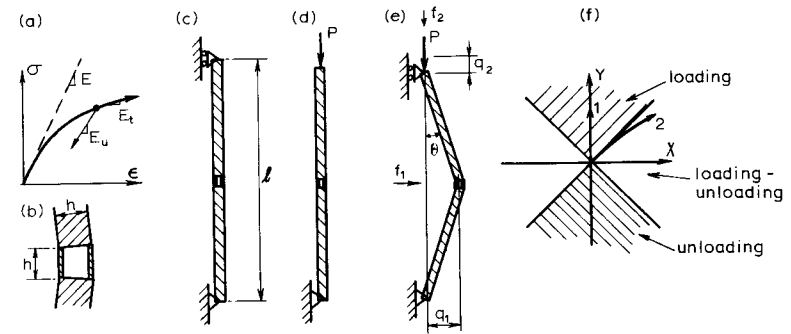


FIG. 1. Shanley's Elastoplastic Column

Initially the column is perfectly straight and is loaded by an axial centric load  $P$  (positive if compressive). The initial equilibrium under load  $P = P_0$  at zero lateral deflection is disturbed by increasing the axial load to  $P = P_0 + \delta f_2$  and applying a small lateral load  $\delta f_1$ . This causes in the flanges of the elasto-plastic link the strains  $\delta \epsilon_1 = -\delta \theta - \delta q_2/h$  and  $\delta \epsilon_2 = \delta \theta - \delta q_2/h$  (on the concave and convex sides, respectively);  $\delta \theta = 2\delta q_1/l =$  rotation of the rigid bars, assumed to be small.

The incremental moduli for loading and unloading are denoted as  $E_l$  and  $E_u$  [Fig. 1(a)];  $E_l$  is called the tangential modulus. Always  $E_l < E_u$  (except when the material is elastic, in which case  $E_l = E_u$ ).  $E_l$  is a given function of the initial uniform stress  $\sigma = \sigma_1 = \sigma_2 = -P/A$ . It is convenient to express the moduli at the concave and convex faces as  $E_1 = \eta E_l$  and  $E_2 = \eta \xi E_l$  and define the nondimensional displacements  $X = \delta q_1/l$  and  $Y = \delta q_2/2h$ . The force variables associated by work are  $f_X = l\delta f_1$  and  $f_Y = 2h\delta f_2$ . Based on the foregoing expressions for strains, one may then obtain for buckling to the right ( $X > 0$ ) the following loading-unloading criteria [Fig. 1(c)]:

$$\text{for } Y > X \text{ (loading only): } \xi = 1, \quad \eta = 1 \dots \dots \dots (1a)$$

$$\text{for } -X \leq Y \leq X \text{ (loading-unloading): } \xi = \xi_u, \quad \eta = 1 \dots \dots \dots (1b)$$

$$\text{for } Y < -X \text{ (unloading only): } \xi = 1, \quad \eta = \xi_u \dots \dots \dots (1c)$$

where  $\xi_u = E_u/E_l$ . Based on the incremental stresses  $\delta \sigma_1 = E_1 \delta \epsilon_1$  and  $\delta \sigma_2 = E_2 \delta \epsilon_2$  at the concave and convex faces, the moment and axial conditions of equilibrium at the midspan lead, for buckling to the right, to the equations:

$$\begin{Bmatrix} l\delta f_1 \\ 2h\delta f_2 \end{Bmatrix} = 2\eta P_l l \begin{bmatrix} 1 + \xi - 2P_0/\eta P_l & 1 - \xi \\ 1 - \xi & 1 + \xi \end{bmatrix} \begin{Bmatrix} X \\ Y \end{Bmatrix} \dots \dots \dots (2)$$

in which  $P_l = E_l A h / l =$  Shanley's tangent modulus load for modulus  $E_l$  (Bažant 1985, 1986; Bruhns and Mielniczuk 1977; Bruhns 1984; Chajes 1974). If  $\delta f_2 = 0$  ( $P =$  constant), then the only nonzero solution of Eq. 2 is  $P_0 = 2\xi_u P_l / (\xi_u + 1) = P_r$ .  $P_r$  represents the reduced modulus load of Engesser and von Kármán (see e.g., Chajes 1974; Chen and Atsuta 1976; Chen and Lui 1987; Johnston 1976; Timoshenko and Gere 1961), at which there is neutral equilibrium [Fig. 2(a)]. For a straight segment of the  $\sigma - \epsilon$  diagram,

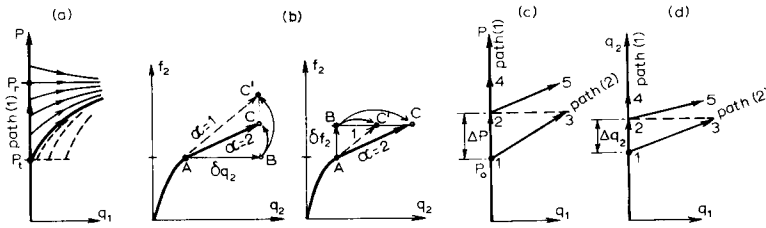


FIG. 2. Bifurcations of Equilibrium Paths

$E_t$ ,  $P_t$ , and  $P_r$  are constant. When  $E_t$  depends on  $\sigma$  (curved  $\sigma - \epsilon$  diagram),  $P_t$  and  $P_r$  depend on  $\sigma_0 = -P/A$  through  $E_t(\sigma_0)$ .

Because the inequalities in Eq. 1 offer several alternatives, Eq. 2 for  $\delta f_1 = 0$  permits more than one equilibrium path. One path, called the primary path or path 1 (Fig. 2), is characterized by zero deflection and  $\xi = \eta = 1$ ;

$$X^{(1)} = 0, \quad \delta f_2^{(1)} = 2P_t \frac{l}{h} Y^{(1)} \dots \dots \dots (3)$$

For  $P_0 \rightarrow P_t$ , however,  $X^{(1)}$  is arbitrary since the matrix of Eq. 2 becomes singular (but  $X^{(1)} < Y^{(1)}$  because  $\xi = 1$ ). Another path, called the secondary path or path 2 (Fig. 2), is characterized by positive deflection (at  $f_1 = 0$ ),  $\eta = 1$  and  $\xi > 1$ . For this path Eqs. 2 and 3 yield

$$X^{(2)} = \frac{\xi_u - 1}{\xi_u + 1 - \frac{2P_0}{P_t}} Y^{(2)}, \quad \delta f_2^{(2)} = \frac{P_t l}{h} \frac{4\xi_u P_t - 2(\xi_u + 1)P_0}{(\xi_u + 1)P_t - 2P_0} Y^{(2)} \dots \dots \dots (4)$$

The superscripts (1) and (2) refer to paths 1 and 2. Path 2 according to Eq. 4 is possible only if  $\xi > 1$ , and  $\xi > 1$  is possible only if  $P_0 \geq P_t$ . Since the solutions in Eqs. 3 and 4 exist for each point  $P_0 \geq P_t$  [solid curves in Fig. 2(a)], the main path for  $P \geq P_t$  represents a continuous sequence of bifurcation points. The first bifurcation occurs at  $P_0 = P_t$ , as shown by Shanley [Fig. 2(a)]. Note that for  $P_0 = P_t$  we have  $Y^{(2)}/X^{(2)} = 1$ , i.e. the secondary path starts along the boundary of the loading-only sector in the  $(X, Y)$  plane.

To analyze stability, we calculate the second-order incremental work  $\delta^2 W$  of small equilibrium forces  $\delta f_1$  and  $\delta f_2$  on arbitrary small incremental displacements  $\delta q_1$  and  $\delta q_2$ . We have  $\delta^2 W = 1/2(\delta f_1 \delta q_1 + \delta f_2 \delta q_2)$ , and substitution of Eq. 2 provides:

$$\delta^2 W(X, Y) = \frac{P_t l \eta}{\xi + 1} \left\{ [(\xi + 1)Y - (\xi - 1)|X|]^2 + 4\xi \left( 1 - \frac{(\xi + 1)P_0}{2\xi P_t \eta} \right) X^2 \right\} \dots \dots (5)$$

The absolute value  $|X|$  is introduced here in order to make Eq. 5 valid for buckling to both right and left.

Eq. 5 can alternatively be derived as  $\delta^2 W = \delta^2 W_0 + \delta^2 W_\sigma$  in which  $\delta^2 W_0 = (1/2)(\delta\sigma_1 \delta\epsilon_1 + \delta\sigma_2 \delta\epsilon_2) hA/2 =$  second-order work of stresses, and  $\delta^2 W_\sigma = -P_0 \Delta l$ , with  $\Delta l = l(1 - \cos \theta) \approx l(\delta\theta)^2/2 = 2lX^2 =$  second-order axial displacement at the load point. Note that  $\delta^2 W_\sigma = (1/2)(G_{11}X^2 + 2G_{12}XY$

$= G_{22}Y^2)$  where  $G_{ij} =$  geometric stiffness matrix; and  $G_{11} = -4P_0 l$ ,  $G_{12} = G_{22} = 0$ .

Eq. 5 applies in general if both the axial load and the column length can change during the incremental deformation. Under a displacement-controlled mode of loading, we have  $\delta q_2 = 0$  during buckling, and then  $\delta^2 W = 1/2 \delta f_1 \delta q_1$ , i.e. (for  $Y \geq 0$ ):

$$\delta^2 W = [\delta^2 W]_{Y=\text{const}} = \delta^2 W(X) = \eta P_t l \left( \xi + 1 - \frac{2P_0}{\eta P_t} \right) X^2 \dots \dots \dots (6)$$

On the other hand, under a load-controlled mode of loading ( $\delta f_2 = 0$ ):

$$\delta^2 W = [\delta^2 W]_{P=\text{const}} = \delta^2 W(X) = \eta P_t l \left( \frac{4\xi}{\xi + 1} - \frac{2P_0}{\eta P_t} \right) X^2 \dots \dots \dots (7)$$

To identify the stable path, we will also need the work  $\delta^2 W$  done along the equilibrium paths (for  $f_1 = 0$ ). For path 2,  $\delta^2 W^{(2)} = 1/2 \delta f_2^{(2)} \delta q_2^{(2)}$ . Substituting Eq. 4 we get, in terms of  $Y$ ,

$$\delta^2 W^{(2)} = \frac{2\xi_u P_t - (\xi_u + 1)P_0}{(\xi_u + 1)P_t - 2P_0} 2P_t l Y^2 \dots \dots \dots (8)$$

and in terms of  $\delta f_2$

$$\delta^2 W^{(2)} = \frac{(\xi_u + 1)P_t - 2P_0}{2\xi_u P_t - \xi_u - 1} \frac{h^2}{2P_t l} \delta f_2^2 \dots \dots \dots (9)$$

For path 1,  $\delta^2 W^{(1)} = 1/2 \delta f_2^{(1)} \delta q_2^{(1)} = 2P_t l Y^2$  in terms of  $Y$ , or  $\delta^2 W^{(1)} = h^2 \delta f_2^2 / 2P_t l$  in terms of  $\delta f_2$ , according to Eq. 3. The work difference is found to be, when  $Y$  is the same,

$$\delta^2 W^{(1)} - \delta^2 W^{(2)} = \frac{(P_0 - P_t)(\xi_u - 1)P_t l}{\left( 1 + \frac{(\xi_u - 1)^2}{4\xi_u} \right) P_t - P_0} Y^2 \dots \dots \dots (10)$$

and when  $\delta f_2$  is the same,

$$\delta^2 W^{(1)} - \delta^2 W^{(2)} = -\frac{(P_0 - P_t)(\xi_u - 1)h^2}{2(P_t - P_0)(\xi_u + 1)P_t l} \delta f_2^2 \dots \dots \dots (11)$$

**GENERAL CRITERION FOR STABILITY OF STATE**

The state of the structure is characterized by displacements  $q_k$  ( $k = 1, 2, \dots, n$ ). The total energy and Helmholtz free energy of the structure are defined as  $U = \mathcal{W} + Q$ ,  $F = U - TS$  where  $d\mathcal{W} = \mathbf{P} \cdot d\mathbf{q} =$  work of loads  $P_1, \dots, P_n$  (applied forces),  $\mathbf{q} = (q_1, \dots, q_n) =$  vector of load-point displacements,  $\mathbf{P} = (P_1, \dots, P_n)$ ,  $T =$  absolute temperature (assumed to be uniform),  $dQ =$  heat that flows into the structure from the outside, and  $S =$  entropy of the structure, which is defined as  $dS = (dQ/T) + dS_{in}$  where  $dS_{in} =$  internal entropy of the structure. From these relations, one can verify that

$$dF = \mathbf{P} \cdot d\mathbf{q} - SdT - TdS_{in} \dots \dots \dots (12)$$

$$dU = \mathbf{P} \cdot d\mathbf{q} + TdS - TdS_{in} \dots \dots \dots (13)$$

Considering isothermal ( $dT = 0$ ) and isentropic ( $dS = 0$ ) infinitesimal equilibrium deformations (which must be path-independent in the small), we have, according to Eqs. 12 and 13,

$$dF = \mathbf{f}_T \cdot d\mathbf{q}, \quad dU = \mathbf{f}_S \cdot d\mathbf{q} \dots \dots \dots (14)$$

where  $\mathbf{f}_T, \mathbf{f}_S$  are the vectors of associated forces (reactions) that depend on  $\mathbf{q}$  according to isothermal or isentropic material properties;  $\mathbf{f}_T = (f_1, \dots, f_n)_T, \mathbf{f}_S = (f_1, \dots, f_n)_S$ . These forces must be distinguished from the applied loads which follow their own law (e.g. gravity) that is independent of the response of the structure.

Because of the principle of conservation of energy,

$$\delta F = \int_V \boldsymbol{\sigma}_T : \delta \boldsymbol{\epsilon} dV, \quad \delta U = \int_V \boldsymbol{\sigma}_S : \delta \boldsymbol{\epsilon} dV \dots \dots \dots (15)$$

where  $V =$  volume of the body,  $\delta \boldsymbol{\epsilon} =$  strain tensor increment,  $\boldsymbol{\sigma}_T, \boldsymbol{\sigma}_S =$  current stress tensors calculated from the history of  $\boldsymbol{\epsilon}$  on the basis of isothermal or isentropic material properties. In the case of geometric nonlinearity or buckling phenomena, only certain types of stress and strain tensors, known from finite-strain continuum mechanics, can be used for  $\boldsymbol{\sigma}$  and  $\boldsymbol{\epsilon}$  in order to ensure equivalence of the work of stresses (Eq. 15) to the work of reaction forces (Eq. 14) (see, e.g., Hill 1958, 1962; Maier 1971).

If the problem is geometrically linear, the equivalence of Eqs. 14 and 15 also follows from the principle of virtual work (Maier 1971; Maier and Drucker 1973) because  $\boldsymbol{\sigma}$  is in equilibrium with  $\mathbf{f}$  and  $\boldsymbol{\epsilon}$  is compatible with  $\mathbf{q}$ . Note that the term  $TdS_{in}$  does not belong in Eqs. 14 and 15 because  $\mathbf{f}_T$  and  $\mathbf{f}_S$  as well as  $\boldsymbol{\sigma}_T$  and  $\boldsymbol{\sigma}_S$  must be evaluated to include both reversible and irreversible responses (see Eqs. 19 and 20 where  $\mathbf{K}_{jk}$  and  $\mathbf{C}_T$  include plastic stiffness). Because  $dS_{in} = 0$  in Eqs. 14 and 15, we have  $dS = dQ/T$ , and so the isentropic ( $dS = 0$ ) and adiabatic ( $dT = 0$ ) material properties are equivalent for the determination of  $\mathbf{f}_S$  and  $\boldsymbol{\sigma}_S$ .

It is convenient to introduce the Helmholtz free energy  $\mathcal{F}$  and the total energy  $\mathcal{Q}$  of the structure-load system;  $\mathcal{F} = F - \mathcal{W}$  and  $\mathcal{Q} = U - \mathcal{W}$ . For an elastic structure,  $\mathcal{F}$  and  $\mathcal{Q}$  reduce to the potential energy. But for an inelastic structure they do not represent potentials in the mathematical sense since they are irreversible, path-dependent variables. According to Eqs. 12–15 we obtain

$$d\mathcal{F} = -SdT - TdS_{in} = \int_V \boldsymbol{\sigma}_T : d\boldsymbol{\epsilon} dV - \mathbf{P} \cdot d\mathbf{q} = \mathbf{f}_T \cdot d\mathbf{q} - \mathbf{P} \cdot d\mathbf{q} \dots \dots \dots (16)$$

$$d\mathcal{Q} = TdS - TdS_{in} = \int_V \boldsymbol{\sigma}_S : d\boldsymbol{\epsilon} dV - \mathbf{P} \cdot d\mathbf{q} = \mathbf{f}_S \cdot d\mathbf{q} - \mathbf{P} \cdot d\mathbf{q} \dots \dots \dots (17)$$

Consider now a change from the initial equilibrium state  $\mathbf{q}^0$  to a neighboring state  $\mathbf{q}^0 + \delta\mathbf{q}$ . Since inelastic response must be path-independent in the small, we may introduce Taylor series expansion of function  $\mathbf{f}(\mathbf{q})$ . Assuming the applied loads  $\mathbf{P}$  to be constant (dead loads) we thus obtain

$$\Delta\mathcal{F} = \int_{q_j^0}^{q_j^0 + \delta q_j} \sum_j \left[ f_j^0 + \sum_k f_{j,k}^0 (q_k - q_k^0) \right] \dots \dots \dots \quad [Continued]$$

$$+ \frac{1}{2} \sum_k \sum_m f_{j,km}^0 (q_k - q_k^0)(q_m - q_m^0) + \dots \dots \dots \left] dq_j - \sum_j P_j \delta q_j \dots \dots \dots (18)$$

where  $f_j^0, f_{j,k}^0, f_{j,km}^0 + \dots$  are the initial values of the equilibrium forces and their partial derivatives with respect to  $q_k$  and  $q_m$ . Because the initial state is an equilibrium state, we have  $\mathbf{f}^0 = \mathbf{P}$ , which means that the first-order work term  $\delta^1\mathcal{W} = \sum_j P_j \delta q_j$  cancels out.

Let us now assume that  $f_{j,k}^0$  are nonzero. Consider that the change of the structure state is isothermal ( $dT = 0$ ). Then, after integrating Eq. 18 up to second-order terms in  $\delta q_k$  and setting  $\sum_k f_{j,k}^0 \delta q_k = \delta f_j =$  equilibrium force changes, we obtain

$$\Delta\mathcal{F} = -TdS_{in} = \frac{1}{2} \delta \mathbf{f} \cdot \delta \mathbf{q} = \sum_j \frac{1}{2} \delta f_j \delta q_j = \sum_j \sum_k \frac{1}{2} K_{jk}(\mathbf{v})_T \delta q_j \delta q_k = \delta^2\mathcal{W} \quad (19)$$

where  $K_{ij}(\mathbf{v})_T$  is the tangential stiffness matrix associated with  $\mathbf{q}$ , evaluated on the basis of isothermal material properties ( $dT = 0$ ). This matrix depends in general on the direction vector  $\mathbf{v}$  of the displacements  $\delta\mathbf{q}$ , as illustrated by Eq. 2 in which the values of  $\xi$  and  $\eta$  depend on the ratio  $Y/X$ .

From the principle of conservation of energy it further follows that

$$\Delta\mathcal{F} = -TdS_{in} = \int_V \frac{1}{2} \delta \boldsymbol{\sigma}_T : \delta \boldsymbol{\epsilon} dV + \delta^2 W_\sigma = \int_V \frac{1}{2} \delta \boldsymbol{\epsilon} : \mathbf{C}_T : \delta \boldsymbol{\epsilon} dV + \delta^2 W_\sigma \dots \dots (20)$$

Here  $\mathbf{C}_T =$  tensor of isothermal tangential moduli of the material, which depends on whether the material loads or unloads;  $\delta^2 W_\sigma = \sum \sum (1/2) \delta q_k G_{km} \delta q_m$ ; and  $G_{km} =$  geometric stiffness matrix, which depends (linearly) on the initial stress  $\sigma_{ij}^0$  (or  $P_0$ ) but is independent of  $\mathbf{C}_T$ . In absence of geometric non-linear effects ( $G_{ij} \approx 0$ ), Eq. 20 also follows from the principle of virtual work, since  $\delta \mathbf{f}$  is in equilibrium with  $\delta \boldsymbol{\sigma}$  and  $\delta \boldsymbol{\epsilon}$  is compatible with  $\delta \mathbf{q}$ .

Similarly, for isentropic deformations ( $dS = 0$ ):

$$\Delta\mathcal{Q} = -TdS_{in} = \frac{1}{2} \delta \mathbf{f} \cdot \delta \mathbf{q} = \sum_j \frac{1}{2} \delta f_j \delta q_j = \sum_j \sum_k \frac{1}{2} K_{jk}(\mathbf{v})_S \delta q_j \delta q_k = \delta^2\mathcal{W} \quad (21)$$

$$\Delta\mathcal{Q} = -TdS_{in} = \int_V \frac{1}{2} \delta \boldsymbol{\sigma}_S : \delta \boldsymbol{\epsilon} dV + \delta^2 W_\sigma = \int_V \frac{1}{2} \delta \boldsymbol{\epsilon} : \mathbf{C}_S : \delta \boldsymbol{\epsilon} dV + \delta^2 W_\sigma \dots \dots (22)$$

Here  $K_{jk}(\mathbf{v})_S$  and  $\mathbf{C}_S$  must be evaluated on the basis of adiabatic (equivalent to isentropic) material properties.

According to the second law of thermodynamics, the changes for which  $\Delta S_{in} > 0$  will occur and those for which  $\Delta S_{in} < 0$  cannot occur. Therefore, the structure is *stable* (i.e. remains in its initial state) if  $\Delta S_{in} < 0$  for all possible  $\delta q_k$  or all possible  $\delta f_k$ . The structure is *unstable* if  $\Delta S_{in} > 0$  for some  $\delta q_k$  or some  $\delta f_k$ . This criterion, whose essence for fluids was stated already by Gibbs, is the fundamental criterion of stability of the state of any system, including structures, whether reversible (elastic) or irreversible (inelastic). According to Eqs. 19–22, this criterion may also be stated as follows:

The structure is *stable* if the second-order work given by Eqs. 19–20 (for  $dT = 0$ ) or Eqs. 20 and 21 (for  $dS = 0$ ) is positive for all possible  $\delta q_k$  (or all possible  $\delta f_k$ ), in other words, if it is positive definite. The structure is *unstable* if this second-order work is negative for some  $\delta q_k$  (or some  $\delta f_k$ ).

The case when the second-order work is zero for some  $\delta q_k$  (or some  $\delta f_k$ ) represents the critical state, defined as the limit of stable states. Whether the critical state is stable or unstable can be decided only on the basis of the higher-order terms of the Taylor series expansion in Eq. 18, which furnishes higher-order terms to be added to Eqs. 19–22.

If the loads  $\mathbf{P}$  are variable, higher-order terms involving  $\mathbf{P}$  need to be added to Eq. 18, and consequently also to Eqs. 19–22. The stability criterion  $\Delta S_{in} < 0$  remains valid whether loads  $\mathbf{P}$  are conservative or nonconservative, but it of course cannot detect dynamic instabilities such as flutter.

The special case of the foregoing criterion of stability of equilibrium states for plastic materials and for boundary conditions of fixed displacements was presented, without thermodynamic derivation, by Hill (1958).

### STABLE EQUILIBRIUM STATES OF ELASTOPLASTIC COLUMN

According to Eq. 19 or 21, the condition of stability is that the expressions in Eq. 5 (for load control) or Eq. 6 (for displacement control) must be positive for all possible  $X$  and  $Y$  (i.e., positive definite). From this we conclude that under load control Shanley's column is stable if  $P_0 < P_{cr}^L$  and unstable if  $P_0 > P_{cr}^L$  where

$$P_{cr}^L = P_r = \frac{2\xi_u}{\xi_u + 1} P_t \dots \dots \dots (23)$$

Under displacement control, the column is stable if  $P_0 < P_{cr}^D$  and unstable if  $P_0 > P_{cr}^D$  where

$$P_{cr}^D = \frac{\xi_u + 1}{2} P_t = \left( 1 + \frac{(\xi_u - 1)^2}{4\xi_u} \right) P_r > P_r \dots \dots \dots (24)$$

For elastic columns, by contrast,  $P_{cr}^L = P_{cr}^D$ . The physical reason for  $P_{cr}^D$  to be higher than  $P_{cr}^L$  is that lateral deflection of an elastoplastic column at constant  $P$  is accompanied by axial displacement (Eq. 4). Note that  $P_{cr}^D$  is the  $P_0$ -value for which Eq. 2 with  $\delta f_1 = Y = 0$  has a nonzero solution (at  $\delta f_2 \neq 0$ ), while  $P_{cr}^L$  is the  $P_0$ -value for which Eq. 2 with  $\delta f_1 = \delta f_2 = 0$  ( $P = \text{constant}$ ) has a nonzero solution.

The main aspects of the present stability problem can be illustrated by the surfaces in Fig. 3. This figure shows (for  $E_u = 3E_t$ ) three-dimensional views of the surfaces of  $\delta^2 W$  or  $T\Delta S_{in} = -\delta^2 W$  given by Eq. 5 as functions of  $X$  and  $Y$ . The equilibrium state is characterized by  $\partial \delta^2 W / \partial X = 0$  and  $\partial \delta^2 W / \partial Y = 0$ , and accordingly, all the surfaces shown have zero slopes at the origin, for any direction.

Eq. 5, which can be written as  $\delta^2 W = \sum 1/2 K_{jk} \delta q_j \delta q_k$ , appears to be a quadratic form but is not, because  $\xi$  and  $\eta$  depend on  $\delta q_2 / \delta q_1$ . The surfaces in Fig. 3 consist of quadratic portions separated by the lines  $X = \pm Y$  at which  $\xi$  or  $\eta$  changes discontinuously. These are lines of curvature discontinuity. Therefore, in contrast to the potential energy surfaces for elastic stability problems (Thompson 1982), the present surfaces are not smooth. They are nevertheless continuous. They also have continuous slopes, because the gradient represents the equilibrium forces ( $f_k = \partial W / \partial q_k$ ). The curvatures, however, must be discontinuous because they represent the incre-

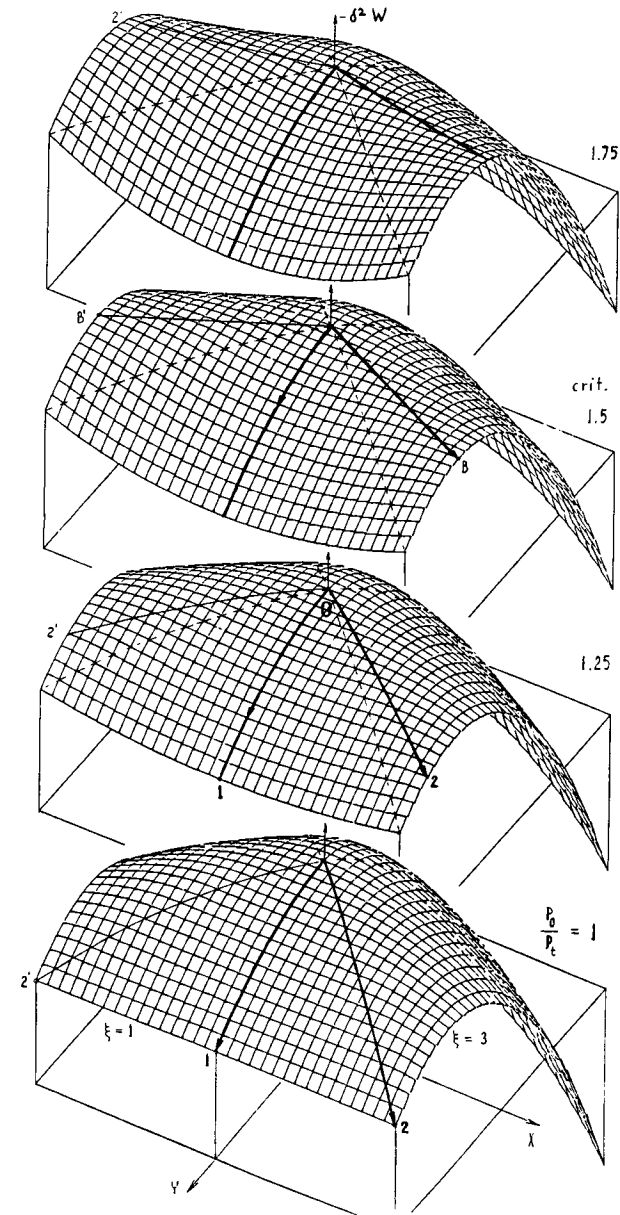


FIG. 3. Surfaces of Internal Entropy Increment  $\Delta S_{in} = -\delta^2 W/T$  for Shanley's Column at Various Load Levels

mental stiffnesses of the structure, which change discontinuously (e.g. from  $E_i$  to  $E_u$ ).

Note that the surfaces of  $\delta^2 W$  describe path-independent behavior only for those paths for which the path direction at all points of the path belongs to the same sector of  $\nu$ -directions in the  $q$ -space. For other paths, there is path-dependence.

The stack of the surfaces for  $P_0/P_i = 1, 1.25, 1.5$  (crit.),  $1.75$  shows how these surfaces evolve as the load is increased. The limit of stable states ( $P_0 = 1.5P_i = P_{cr}$ ) is manifested on these surfaces by the existence of a horizontal path emanating from the origin 0 (lines  $B$  or  $B'$  in the figures). Instability is characterized by the existence of a path for which  $\delta^2 W$  descends (or  $-\delta^2 W$  rises) while moving away from the origin. Absence of any such path ensures stability. Note that for  $P_0/P_i = 1.25$  a portion of the surface in Fig. 3 for  $Y > |X|$  is a hyperbolic paraboloid, even though the state is stable.

### GENERAL CRITERIA OF STABLE EQUILIBRIUM PATH

It has often been alleged that the critical load of elastoplastic columns is between  $P_r$  and  $P_i$ , and  $P_i$  has sometimes been called the critical load. As we now see from Eqs. 23 and 24,  $P_i$  is not the critical load but merely the lowest bifurcation load. Doesn't this conflict with Shanley's proposition that columns should be designed for the tangent modulus load  $P_i$ ? Why should perfect columns start to deflect at  $P_i$  rather than remaining straight up to  $P = P_{cr}$ ?

To answer these questions, we note that an equilibrium path represents a series of infinitesimal deviations from equilibrium and its restorations. Therefore, we consider, for an arbitrary structure, a small loading step along the equilibrium path  $\alpha$  ( $\alpha = 1$  or  $2$ ) which starts at the bifurcation state  $A$  (at which the applied load is  $P_0$ , and  $X = 0$ ) and ends at another state  $C$  on path  $\alpha$  [Fig. 2(b)]. We decompose this step into two substeps, the first one (I) away from the initial equilibrium state  $A$  and ending at some intermediate nonequilibrium state  $B$ , and the second one (II) toward a new equilibrium state, ending on one of the equilibrium paths at state  $C$ ; see Fig. 2(b). (This decomposition was introduced in Bažant (1985), which presented a special case of the analysis that follows.) The displacements or forces which are controlled are denoted as  $q_m, f_m$ . For our column, only one displacement or load is controlled, namely  $q_2$  or  $f_2$ , i.e.  $m = 2$ . If  $q_m$  is controlled, we consider  $\delta q_m$  to be changed in the first substep [Fig. 2(b), left] while  $f_m$  are kept constant, which of course destroys equilibrium;  $\delta q_m$  are kept constant during the second substep in which  $\delta f_m$  are allowed to change so as to regain equilibrium. If  $f_m$  are controlled, we consider  $\delta f_m$  to be changed in the first substep [Fig. 2(b), right] while  $q_m$  are frozen (constant), which destroys equilibrium;  $\delta f_m$  are kept constant during the second substep in which  $\delta q_m$  are allowed to change so as to restore equilibrium.

First consider displacement control (i.e.,  $\delta q_m$  prescribed) and isothermal conditions ( $dT = 0$ ). Then  $\delta q_m$  are the same for all the paths  $\alpha$  but the equilibrium force increments  $\delta f_m^{(\alpha)}$  are different. According to Eq. 15, the increment of Helmholtz free energy of the structure over the entire step is, up to second-order terms,  $\Delta F = \sum (f_m^0 + 1/2 \delta f_m^{(\alpha)}) \delta q_m = \Delta F_I + \Delta F_{II}^{(\alpha)}$ . Here  $\Delta F_I = \sum f_m^0 \delta q_m =$  increment of  $F$  over the first substep, which is the same for both paths  $\alpha = 1, 2$ , and  $\Delta F_{II}^{(\alpha)} = \sum 1/2 \delta f_m^{(\alpha)} \delta q_m = \delta^2 W^{(\alpha)}$  = increment

of  $F$  in the second substep in which  $q_m$  are constant, while the forces  $f_m$  change by  $\delta f_m^{(\alpha)}$  to find new equilibrium on path  $\alpha$ ;  $\delta^2 W^{(\alpha)}$  is the second-order work along path  $\alpha$ . (For our column, there is only one term in the sum,  $m = 2$ , i.e.  $\Delta F_{II}^{(\alpha)} = 1/2 \delta f_2^{(\alpha)} \delta q_2$ .) According to Eq. 12 with  $dT = 0$ , we have for the second substep (in which  $\delta q = 0$ )  $\Delta F_{II}^{(\alpha)} = -T \Delta S_{in}$ . The second law of thermodynamics indicates that the structure will approach the equilibrium state for which  $\Delta S_{in}$  is maximized (Guggenheim 1959), i.e.  $\Delta F_{II}^{(\alpha)}$  is minimized. Hence, the path  $\alpha$  which actually occurs is that for which

$$-T \Delta S_{in} = \delta^2 W^{(\alpha)} = \sum_m \frac{1}{2} \delta f_m^{(\alpha)} \delta q_m = \sum_m \sum_j \frac{1}{2} K_{mj}^{(\alpha)} \delta q_m \delta q_j = \min \dots \dots \dots (25)$$

(if  $q_m$  is controlled).  $K_{mj}^{(\alpha)}$  is the tangential stiffness matrix for path  $\alpha$ , which must be based on isothermal incremental material properties.

Second consider displacement control ( $\delta q_m$  prescribed) and isentropic conditions ( $dS = 0$ ). According to Eq. 15, the increment of the total energy of the structure over the entire step is, up to the second-order terms,  $\Delta U = \sum (f_m^0 + 1/2 \delta f_m^{(\alpha)}) \delta q_m = \Delta U_I + \Delta U_{II}^{(\alpha)}$ . Here  $\Delta U_I = \sum f_m^0 \delta q_m$ , which is the same for both paths  $\alpha$ , and  $\Delta U_{II}^{(\alpha)} = \sum 1/2 \delta f_m^{(\alpha)} \delta q_m = \delta^2 W^{(\alpha)}$  = increment of  $U$  in the second substep in which  $q_m$  are constant while  $f_m$  change by  $\delta f_m^{(\alpha)}$  to find new equilibrium on path  $\alpha$ . According to Eq. 13 with  $dS = 0$ , we have for the second substep (in which  $\delta q = 0$ )  $\Delta U_{II}^{(\alpha)} = -T \Delta S_{in}$ . The second law of thermodynamics indicates that on approach to equilibrium  $\Delta U_{II}^{(\alpha)}$  must be minimized (Guggenheim 1959). Hence, the path which occurs is again determined by Eq. 25, in which however  $K_{mj}^{(\alpha)}$  must be based on isentropic (adiabatic) rather than isothermal material properties.

Third, consider load control and isothermal conditions. The proper thermodynamic function is now Gibbs' free energy, which is defined as  $G = F - P \cdot q$ . From Eq. 12 one obtains

$$dG = -q \cdot df - SdT - TdS_{in} \dots \dots \dots (26)$$

Here  $q \cdot df$  represents the complementary work. In terms of equilibrium displacements  $q_{rk}$  calculated from isothermal incremental material properties, we have (according to Eq. 26 for  $dT = dS_{in} = 0$ )  $dG = -q_r \cdot df$ . Accordingly we have, for both substeps combined,  $\Delta G = -\sum (q_m + 1/2 \delta q_m^{(\alpha)}) \delta f_m = \Delta G_I + \Delta G_{II}$ . Here  $\Delta G_I = -\sum q_m \delta f_m =$  increment of  $G$  over the first substep, which is the same for both paths  $\alpha = 1, 2$ , and  $\Delta G_{II}^{(\alpha)} = -\sum 1/2 \delta f_m^{(\alpha)} \delta q_m = -\delta^2 \bar{W}^{(\alpha)}$  = increment of  $G$  over the second substep in which  $f_m$  are constant while  $q_m$  are allowed to change so as to restore equilibrium by reaching path  $\alpha$ ;  $\delta^2 \bar{W}^{(\alpha)}$  is the second-order complementary work along path  $\alpha$ . (For our column, there is only one term in the sum,  $m = 2$ , i.e.  $\Delta F_{II}^{(\alpha)} = -1/2 \delta f_2^{(\alpha)} \delta q_2$ .) According to Eq. 26 with  $dT = 0$ , we have for the second substep (in which  $\delta f = 0$ )  $\Delta G_{II}^{(\alpha)} = -T \Delta S_{in}$ . Based on the second law of thermodynamics, the approach to new equilibrium must maximize  $\Delta S_{in}$ , i.e. minimize  $\Delta G_{II}^{(\alpha)}$  (Guggenheim 1959). Hence, the path  $\alpha$  which actually occurs is that for which

$$T \Delta S_{in} = \delta^2 \bar{W}^{(\alpha)} = \sum_m \frac{1}{2} \delta f_m \delta q_m^{(\alpha)} = \sum_m \sum_j \frac{1}{2} D_{mj}^{(\alpha)} \delta f_m \delta f_j = \max \dots \dots \dots (27)$$

(if  $f_m$  is controlled).  $D_{mj}^{(\alpha)}$  is the tangential compliance matrix for path  $\alpha$ , which must be based on isothermal material properties. Note that, in contrast to

Eq. 25, the path label ( $\alpha$ ) now appears with  $\delta q_m$  rather than  $\delta f_m$ .

Fourth, consider load control and isentropic conditions ( $dS = 0$ ). The proper thermodynamic function is now the enthalpy, which is defined as  $H = U - \mathbf{P} \cdot \mathbf{q}$ . From Eq. 13 one obtains

$$dH = -\mathbf{q} \cdot d\mathbf{f} + TdS - TdS_m \dots \dots \dots (28)$$

In terms of equilibrium displacements  $\mathbf{q}_s$  calculated from isentropic (adiabatic) incremental material properties we have (according to Eq. 28 for  $dS = dS_m = 0$ )  $dH = -\mathbf{q}_s \cdot d\mathbf{f}$ . Thus, for both substeps combined,  $\Delta H = -\sum(q_n + 1/2 \delta q_n^{(\alpha)})\delta f_n = \Delta H_I + \Delta H_{II}$ . Here  $\Delta H_I = -\sum q_m \delta f_m =$  increment of enthalpy  $H$  over the first substep, which is the same for both paths  $\alpha = 1, 2$ , and  $\Delta H_{II}^{(\alpha)} = -\sum 1/2 \delta f_m^{(\alpha)} \delta q_m = -\delta^2 \bar{W}^{(\alpha)}$  = increment of  $H$  over the second substep in which  $f_m$  are constant while  $q_m$  are allowed to change so as to restore equilibrium. According to Eq. 28 with  $dS = 0$ , we have for the second substep (in which  $\delta \mathbf{f} = \mathbf{0}$ )  $\Delta H_{II}^{(\alpha)} = -T\Delta S_m$ . In view of the second law, the approach to new equilibrium must maximize  $\Delta S_m$ , i.e. minimize  $\Delta H_{II}^{(\alpha)}$ . Hence, the path which occurs is again that indicated by Eq. 27, in which however  $D_m^{(\alpha)}$  must be based on isentropic rather than isothermal material properties.

Eq. 20 further shows that, along path  $\alpha$ ,

$$\delta^2 \bar{W}^{(\alpha)} = \sum_m \frac{1}{2} \delta f_m^{(\alpha)} \delta q_m = \int_V \frac{1}{2} \delta \boldsymbol{\sigma}^{(\alpha)} : \delta \boldsymbol{\epsilon}^{(\alpha)} dV + \sum_m \sum_m \frac{1}{2} \delta \mathbf{q}_k G_{km} \delta \mathbf{q}_m \dots \dots (29)$$

where  $\delta \boldsymbol{\sigma}^{(\alpha)}$ ,  $\delta \boldsymbol{\epsilon}^{(\alpha)}$  = stress and strain tensor increments along path  $\alpha$ ; and  $G_{km}$  = geometric stiffness matrix. For  $\delta^2 \bar{W}^{(\alpha)}$ , a similar expression generally does not exist because matrix  $G_{km}$  may be (and often is) singular. But in absence of nonlinear geometric effects ( $G_{km} = 0$ ),

$$\delta^2 \bar{W}^{(\alpha)} = \sum_m \frac{1}{2} \delta f_m \delta q_m^{(\alpha)} - \int_V \frac{1}{2} \delta \boldsymbol{\sigma}^{(\alpha)} : \delta \boldsymbol{\epsilon}^{(\alpha)} dV \dots \dots \dots (30)$$

In finite element programs, the sums of  $\delta f_m \delta q_m$  are usually more efficient to calculate than the volume integrals.

It may also happen that some displacements,  $q_i^{(\alpha)}$  ( $i = 1, \dots, m$ ), and some loads,  $f_j^{(\alpha)}$  ( $j = m + 1, \dots, n$ ), are controlled. For such mixed control, Legendre transformation may then be used to introduce a semi-complementary thermodynamic function,  $Z = F - \sum_j f_j^{(\alpha)} q_j$ . Then, for isothermal conditions,

$$dZ = -SdT + TdS_m + \sum_i f_i^{(\alpha)} dq_i - \sum_j q_j^{(\alpha)} df_j \dots \dots \dots (31)$$

A procedure similar to that which led to Eq. 25 (see Bažant 1988) then shows that the stable path is that for which

$$T\Delta S_m = \sum_i \frac{1}{2} \delta f_i \delta q_i^{(\alpha)} - \sum_j \frac{1}{2} \delta f_j^{(\alpha)} \delta q_j = \max_{(\alpha)} \dots \dots \dots (32)$$

among all the paths ( $\alpha$ ). The same result is obtained for isentropic conditions using the mixed thermodynamic function  $\bar{Z} = U - \sum_j f_j^{(\alpha)} q_j$ .

**ANALYSIS OF STABLE EQUILIBRIUM PATH**

Inspecting Eqs. 10 and 11 for Shanley's column, we now find that under displacement control (same  $Y$ ) we always have  $\Delta W^{(2)} < \Delta W^{(1)}$  if  $P_0 > P_c$ ,

and under load control (same  $\delta f_2$ ), we always have  $\Delta W^{(2)} > \Delta W^{(1)}$  if  $P_0 > P_c$ . This means that, for  $P_0 > P_c$ , path 2 must occur and is, therefore, stable, while path 1 cannot occur and is, therefore, unstable. So the column must deflect for  $P_0 > P_c$ . Shanley's load  $P_c$  represents the maximum load of an undeflected column which can be achieved in a continuous loading process, provided  $E_c$  varies continuously.

What is then the meaning of the stable states of a perfect column for  $P_0 > P_c$ ? They can be reached if temporary restraints are placed at the sides of the column to prevent it from buckling. The load may then be raised up to some value  $P_0 > P_c$ . If  $P_0 < P_{cr}^D$  at axial displacement control, or if  $P_0 < P_{cr}^L$  at axial load control, this column will not deflect when the lateral restraint is removed (provided that the column is perfect, of course). So the column is stable at such a load because the initial state does not change. Deflection occurs only if the load is increased further. If  $E_c$  decreases discontinuously (e.g. if the  $\delta$ - $\epsilon$  diagram is bilinear or if the temperature suddenly increases), an undeflected equilibrium state  $P_0 > P_c$  can be reached even without lateral restraints.

The equilibrium paths leading away from the origin are marked in Fig. 3 as 1, 2 and 2'. In the plots of  $\Delta W$ , the structure follows the path that descends steeper with respect to  $Y$  (and less steeply for the plots of  $-\Delta W$ ). The limit of stability of the main path is characterized by the fact that points 1 and 2 (of equal  $Y$ ) are at equal altitude (Fig. 3). This happens on the surfaces for  $P_0/P_c = 1$ . Instability of the main path is characterized by the fact that point 2 in the plots of  $-\delta^2 W$  lies at a higher altitude than point 1 (and a lower altitude for the plots of  $\delta^2 W$ ). (Note also that the states on the main path  $\bar{0}\bar{1}$  in Fig. 3 cannot be called metastable because for  $P_0 > P_c$  it is not possible to move from point 1 to point 2.)

The static structural stability studies in the literature, even those conducted in the most general sense of catastrophe theory (e.g., Thompson 1982), have so far been confined to elastic structures which possess a potential. For inelastic structures, we have two crucial differences: (1) While the surface of the elastic potential is always smooth, the surface of  $\Delta W$  is unsmooth, exhibiting lines of curvature discontinuity; and (2) while the elastic potential surface is path-independent, the surface of  $\Delta W$  applies only to the radial outward paths (i.e.,  $Y/X = \text{constant}$ ) and some paths close to them.

As we have seen, the irreversible systems have two striking properties:

1. The first bifurcation point on the equilibrium path of an inelastic structure does not have to represent the limit of stability, i.e., the states on all the branches emanating from the bifurcation point can be stable (which cannot occur in elasticity).
2. Yet at the same time, the stable states on one branch beyond the first bifurcation point cannot be reached by a continuous loading process.

The basic cause for this behavior lies in the irreversibility of inelastic deformation. As illustrated for Shanley's column, after bifurcation at point 1 in Figs. 2(c) and (d), a subsequent prescribed increment of either axial load  $P$  or axial displacement  $q_2$  can occur along two distinct equilibrium paths leading to points 2 and 3 (actually, if buckling to the left is also considered, there is also a third path  $\bar{1}\bar{3}'$  but it need not be analyzed since it is symmetric to path  $\bar{1}\bar{3}$ ). This is similar to elastic bifurcation. However, contrary to elastic bifurcation, the structure cannot move along path  $\bar{2}\bar{3}$ , not even in a non-

equilibrium (dynamic) manner, and cannot reach at point 3 the same values of  $q_1$ ,  $q_2$  and  $P$ . The cause is the irreversibility (path-dependence) of plastic strain, which prohibits reaching the same values of  $q_1$ ,  $q_2$ ,  $P$  as those reached along path  $\overline{12}$ .

An elastic structure, though, can move along path  $\overline{23}$  in a nonequilibrium manner, and it does reach at point 3 the *same*  $q_1$ ,  $q_2$  and  $P$  values as does path  $\overline{12}$ . This is dictated by path-independence of elastic deformation. If the structure were elastic (reversible, path-independent) then admissibility of path  $\overline{23}$  would cause the potential energy at point 2 to be non-positive definite. For an inelastic structure, on the other hand, the state  $(q_1, q_2, P)$  at point 3 [Figs. 2(c) and 2(d)] cannot manifest itself in the incremental work expression at point 2 since path 2 is kinematically inadmissible. It is for this reason that point 2 in Figs. 2(c) and 2(d) can be stable for inelastic structures but never for elastic structures.

For the same reason, point 2, even if it is infinitely close to point 1, can be a bifurcation state itself, permitting as the subsequent equilibrium paths both path  $\overline{24}$  and path  $\overline{25}$  in Figs. 2(c) and 2(d). Bifurcation states infinitely close to each other, occupying a continuous path (such as  $\overline{124}$ ), are impossible for elastic structures (reversible systems).

The foregoing results force us to broaden the general concept of stability by distinguishing between: (1) A stable equilibrium *state*; and (2) a stable equilibrium *path*.

The stable path is that which (1) Consists entirely of stable states; and (2) maximizes  $\Delta S$ , compared to all other parts. So it is a narrower concept than a stable state.

For elastic (reversible) structures, both concepts are equivalent, and so this distinction does not exist. For irreversible systems, however, an equilibrium state can be stable while the equilibrium path on which it lies may be unstable. This stable state cannot, in reality, be reached. For such systems, examination of stable states is obviously insufficient.

Note also that stability of the state is decided on the basis of deviations away from equilibrium, while stability of the path is decided on the basis of approaches toward equilibrium.

The concept of a stable path does not quite fit the general definition of stability of solutions, as stated in the dynamic definition of stability in the sense of Poincaré and Liapunov (Leipholz 1970; Ziegler 1968). If an infinitely small disturbance (such as lateral load  $f_i$ ) is introduced at the first bifurcation point [point 1 or 2 of Figs. 2(c) and 2(d)], it does not change path  $\overline{124}$  to path  $\overline{13}$  or  $\overline{125}$ ; rather it excludes path  $\overline{124}$  from the paths  $\overline{124}$ ,  $\overline{13}$ ,  $\overline{125}$  which are possible in absence of any disturbance. Thus, instability of a path is not manifested by the creation of a second, distinct path, as a consequence of an infinitely small disturbance. It is manifested by the opposite, namely by the *exclusion* of one or two possible paths.

## CRITERION OF PATH BIFURCATION

Consider now an arbitrary structure with an  $n$ -dimensional column matrix of displacements,  $\mathbf{q} = (q_1, \dots, q_n)$ . Let  $L$  be the loading-only sector,  $U$  the adjacent loading-unloading sectors, and  $\mathbf{K}^L$ ,  $\mathbf{K}^U$  the corresponding  $\mathbf{K}$ . If there are two paths under load control, then  $\mathbf{K}^L \delta \mathbf{q}^{(1)} = \delta \mathbf{f}$  and  $\mathbf{K}^U \delta \mathbf{q}^{(2)} = \delta \mathbf{f}$  where  $\delta \mathbf{f}$  is given. The direction  $\mathbf{v}^{(1)}$  of  $\delta \mathbf{q}^{(1)}$  always lies in sector  $L$ . Prior to the first

bifurcation, the direction  $\mathbf{v}^{(2)}$  of  $\delta \mathbf{q}^{(2)}$  lies outside the corresponding sector  $U$  for all possible sectors  $U$ , i.e., no path 2 exists. After the first bifurcation,  $\mathbf{v}^{(2)}$  lies within the corresponding sector  $U$  at least for one  $U$ , and then path 2 exists.

Suppose now that the tangential material properties vary continuously along the loading path. Then the direction  $\mathbf{v}^{(2)}$  should also vary continuously. So, at first bifurcation, the direction  $\mathbf{v}^{(2)}$  must coincide with the boundary of sector  $L$  (as illustrated by Shanley's column). But then we must have not only  $\mathbf{K}^U \delta \mathbf{q}^{(2)} = \delta \mathbf{f}$  but also  $\mathbf{K}^L \delta \mathbf{q}^{(2)} = \delta \mathbf{f}$ . Subtracting this from  $\mathbf{K}^L \delta \mathbf{q}^{(1)} = \delta \mathbf{f}$  we get  $\mathbf{K}^L (\delta \mathbf{q}^{(2)} - \delta \mathbf{q}^{(1)}) = \mathbf{0}$  where  $\delta \mathbf{q}^{(2)} \neq \delta \mathbf{q}^{(1)}$ . Consequently, the first bifurcation is indicated by singularity of matrix  $\mathbf{K}^L$ , i.e., by the fact that  $\det \mathbf{K}^L = 0$ , or that the smallest eigenvalue  $\lambda_1$  of matrix  $\mathbf{K}^L$  vanishes. This is the well-known condition of Hill (1961, 1962), and the solid corresponding to matrix  $\mathbf{K}^L$  for which multiple solutions exist is called the linear comparison solid. If, however, the tangential material properties change along the loading path discontinuously, then the first bifurcation occurs when the value of  $\lambda_1$  jumps from positive to negative, without  $\mathbf{K}^L$  ever becoming singular.

The eigenvector  $\mathbf{q}^*$  of the singular matrix  $\mathbf{K}^L$  at the first bifurcation can lie either inside or outside sector  $L$ . If  $\mathbf{q}^*$  lies inside  $L$ , then there exists path 2 such that  $\mathbf{K}^L \delta \mathbf{q}^{(2)} = \mathbf{0}$  where  $\delta \mathbf{q}^{(2)} \sim \mathbf{q}^*$ . This means that there is neutral equilibrium, which represents the limit point instability (or snap-through). If  $\mathbf{q}^*$  lies outside sector  $L$  (which is the case for Shanley's column), then  $\delta \mathbf{q}^{(2)}$  cannot coincide with  $\mathbf{q}^*$  but must lie at the boundary of sector  $L$ ; then  $\mathbf{K}^L \delta \mathbf{q}^{(2)} = \delta \mathbf{f}$  where  $\delta \mathbf{f}$  is nonzero. This means that the secondary path at the first bifurcation occurs at increasing load, which represents the Shanley-type bifurcation.

If matrix  $\mathbf{K}^L$  has a negative eigenvalue  $\lambda_1$ , we have  $(\mathbf{K}^L - \lambda_1 \mathbf{I}) \mathbf{q}^* = \mathbf{0}$  where  $\mathbf{I} =$  unit matrix. It follows that  $\delta^2 W \sim \mathbf{q}^{*T} \hat{\mathbf{K}}^L \mathbf{q}^* = \mathbf{q}^{*T} \hat{\lambda}_1 \mathbf{I} \mathbf{q}^* = \lambda_1 \mathbf{q}^{*T} \mathbf{q}^* < 0$ . But this does not imply instability of state if the associated eigenvector  $\mathbf{q}^*$  lies outside  $L$  (which has been ignored in some recent papers). However, the existence of negative  $\lambda_1$  means that a bifurcation point must have been passed and that the state might not lie on a stable path.

If one displacement, say  $\delta q_n$ , is controlled and  $\delta f_1 = \dots = \delta f_{n-1} = 0$ , one may take the foregoing case of load control for which  $\mathbf{K}^L$  is singular at the first bifurcation point, and then scale  $\delta f_n$  and  $\delta q_n$  by a common factor so as to make  $\delta q_n$  for both paths mutually equal. Since such a scaling does not change the eigenvalues of  $\mathbf{K}^L$ , the condition  $\det \mathbf{K} = 0$  also characterizes the first bifurcation point under displacement control (provided the tangential properties vary continuously).

## IMPERFECTIONS AND SYMMETRY

The fact that real columns must start to deflect at  $P_i$  can be independently proven by analyzing the effect of imperfections. The simplest case of imperfect behavior of Shanley's column can be obtained by assuming that (1) The axial load  $P$  is raised up to the value  $P = P_0 = P_i$  while  $f_1 = 0$ ; (2) then a certain small lateral load  $f_1$ , representing the imperfection, is applied at constant  $P$ ; and (3) then the axial load  $P$  is raised further while keeping  $f_1$  constant. For this loading, and for the case that  $\xi = \xi_u =$  constant during deflection (bilinear stress-strain diagram) and  $\eta = 1$ , Eqs. 2 and 3 yield (after elimination of  $Y$ ) the solution:

$$P(q_1) = \frac{P_r(\xi_u - 1)h - \frac{1}{2}f_1l(\xi_u + 1) + 4P_r\xi_u q_1}{(\xi_u - 1)h + 2(\xi_u + 1)q_1} \dots (33)$$

The curves  $P(q_1)$  are plotted for various values of the imperfection force in Fig. 2(a) as the dashed curves. It may now be easily checked that the limit of the curve  $P(q_1)$  for  $f_1 \rightarrow 0$  is Shanley's equilibrium deflection curve of perfect column 1 which begins at load  $P_r$ . Since the deflection curve of an imperfect column converges to the stable path (i.e. path 2) rather than to some unstable path, and since extremely small lateral disturbing loads are inevitable, the column must start to deflect at  $P = P_r$ , same as we concluded it from our analysis of stability. Similar conclusions have been reached before on the basis of numerical solutions for various other types of imperfections and other types of columns.

The bifurcations that we have illustrated correspond to a breakdown of symmetry. The perfect column has the symmetric choice of deflecting either left or right, but once it has deflected to the right it no longer has the choice of deflecting to the left, i.e., its symmetry has broken down. Structures without symmetry do not exhibit equilibrium path bifurcations. This is for example the case of our column if it is perturbed by a lateral load (Eq. 29). Symmetry of any system can be eliminated by introducing suitable imperfections. Does this render the preceding stability analysis useless? Hardly. Imperfect systems are in general harder to solve than the perfect (symmetric) systems, especially since in principle all the possible imperfections have to be considered. (In Eq. 33 we considered only one type of imperfection.)

#### LOCALIZATION OF SOFTENING DAMAGE

Path bifurcations frequently occur in structures which are destabilized by material softening (or damage, fracture). Consider a uniaxially loaded (tensioned or compressed) specimen of length  $L = 2h$ . The value of axial displacement  $u$  is controlled [Fig. 4(a)] and stress  $\sigma$  is the response. The material is assumed to follow the stress-strain curve  $\overline{012345}$  for loading and curves such as  $\overline{16}$ ,  $\overline{27}$ ,  $\overline{38}$ ,  $\overline{49}$  for unloading [Fig. 4(b)]. For the post-peak strain values, there always exists a multiplicity of solutions. The strain  $\epsilon$  can either increase uniformly or localize, increasing in one segment of length  $h$  (loading) and decreasing in another segment of length  $L - h = h$  (unloading) in such a manner that stress  $\sigma$  in both segments remains the same [Fig. 4(a)].

We will assume that, due to material inhomogeneities, the strain cannot localize into a segment shorter than  $h$  where length  $h$  is a material constant. Fig. 4(c) shows various possible equilibrium paths of  $\sigma$  versus average strain  $\bar{\epsilon} = u/2h$ . Path  $\overline{1234}$  is the main path ( $\alpha = 1$ ) for which the strain is uniform, i.e.,  $\epsilon = \bar{\epsilon}$ . After the peak point, there exist infinitely many possible bifurcation points, such as point 2 [Fig. 3(c)], at which the strain localization begins. The response curves for localizations with unloadings which begin at various points may be constructed by averaging the horizontal distances  $c$  and  $d$  or  $e$  and  $f$  as shown in Figs. 4(b) and (c).

The condition of stable equilibrium of the specimen, which was derived in Bažant (1976; see also Bažant 1985, 1986) and coincides with the general stability condition  $\Delta W > 0$  derived here, indicates the limit of stable equilibrium states to be the snapback point. On the path  $\overline{01234}$  correspondin-

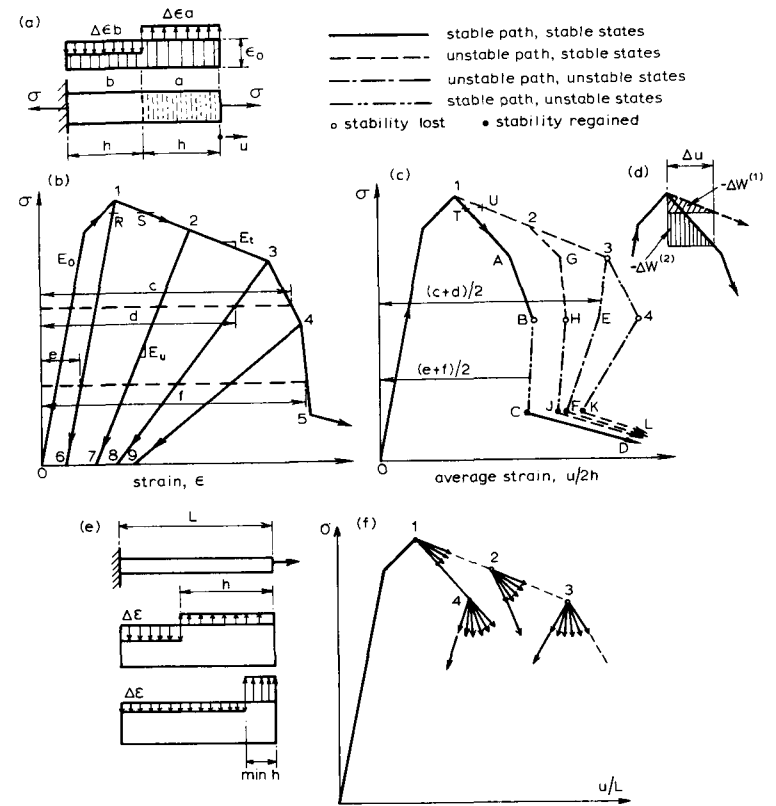


FIG. 4. Bifurcations of Equilibrium Path in Strain-Softening Uniaxial Test Specimen

to uniform strain, it is the point 3. The states of uniform strain on the segment  $\overline{123}$  are stable, despite strain-softening (Bažant 1976, 1977). The limit of stable states (point 3), however, is not the first bifurcation point on the main path. Rather it is the peak point 1. This is similar to the situation for Shanley's column [see the bifurcations in Fig. 2(a)]. The limit of stable states, i.e., the snapback point, is obviously analogous to the reduced modulus load. Does the peak stress point (point 1) correspond to Shanley's tangent modulus load? Does the stable path bifurcate at this point?

The answer is yes. This can be proven by considering the second-order work for prescribed  $\Delta u$ , which is represented by the areas  $\Delta W^{(1)}$  and  $\Delta W^{(2)}$  cross-hatched in Fig. 4(d). Their values are negative, and the smaller one corresponds to the stable path, as we have shown. Since  $\Delta W^{(2)} < \Delta W^{(1)}$ , we must conclude that the stable path bifurcates right at the peak stress point. The specimen, in a continuous loading, must therefore follow path  $\overline{01ABCD}$  in Fig. 4(c), and not the path  $\overline{013E}$ . The stable states on the segment  $\overline{123}$  are not reachable by a continuous loading process. (They would be reachable, though, if the specimen were temporarily forced to deform uniformly, e.g., by first gluing it to a stiff parallel steel plate, then moving



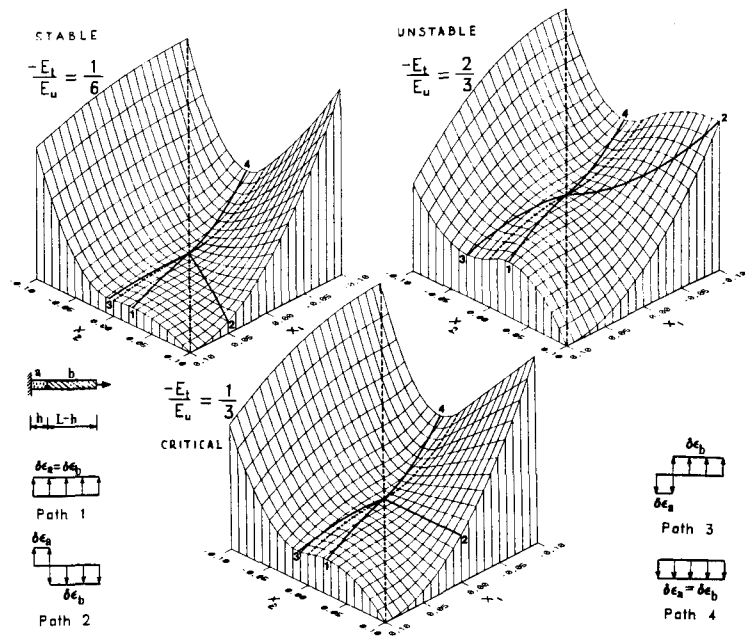


FIG. 5. Surfaces of  $\delta^2 W$  for Strain-Softening Uniaxial Specimen ( $X_1 = \delta u/L$ ,  $X_2 = \delta v/L$ )

the specimen ends and finally dissolving the glue while the ends are kept fixed; the specimen would then remain stable in a state such as point 2.)

To sum up, the thermodynamic criterion of stable path implies that localization of strain begins right after the peak point and the strain always remains localized to the shortest possible segment allowed by the nonlocal characteristics of the material.

The second-order work for the specimen in Fig. 4 is obtained as

$$\delta^2 W = \frac{1}{2} E_u \left[ \frac{\xi}{h} \delta u + \frac{\eta}{L-h} (\delta u - \delta v)^2 \right] \dots \dots \dots (34)$$

in which  $\xi = 1$  for  $\delta v \leq 0$ ,  $\xi = -k$  for  $\delta v > 0$ ,  $\eta = 1$  for  $\delta u \leq \delta v$ ,  $\eta = -k$  for  $\delta u > \delta v$ ;  $k = -E_t/E_u$ ;  $u, v =$  axial displacements at the end and at the interface of segments  $h$  and  $L - h$ . The surfaces of  $\delta^2 W$  are plotted in Fig. 5 for  $L/h = 4$ , and  $k = 1/6, 1/3$ , and  $2/3$ , representing stable, critical and unstable states if  $v$  is controlled; 1, 2, 3, 4 label the paths, and the lines of curvature discontinuity are shown dashed. All the foregoing conclusions can be derived from these surfaces. Note that for stability of state at controlled displacement only positive definiteness of the cross section  $u = 0$  matters, while for stability of path at controlled displacement the positive definiteness of the entire surface of  $\delta^2 W$  matters.

Independently of thermodynamics, our result can also be proven by considering imperfections. Suppose that, for example, a specimen at peak stress state [point 1 in Fig. 4(c)] is subjected, at constant  $u$ , to a very small dis-

turbing axial load  $\delta f$  applied at the interface between the segments  $h$  in Fig. 4(a). This causes one segment to strain-soften to point  $S$  and the other segment to unload to point  $R$  [Fig. 4(b)]. As a result, the state in Fig. 4(c) moves from point 1 to point  $T$  (not to point  $U$ ). Now a crucial property is that for the subsequent loading there is no bifurcation; the segments  $h$  can move in Fig. 4(b) from point  $S$  only toward point 2 and from point  $R$  only toward point 6, which means that the state in Fig. 4(c) moves from point  $T$  toward point  $A$  and cannot move toward point 2. Since an infinitesimal disturbance  $\delta f$  is inevitable, the bifurcated path  $\bar{1}T\bar{A}$  must be followed.

The length of the localization zone can in reality have any value between  $\max h = L$  and  $\min h = l$  where  $l$  is a material constant proportional to characteristic length. For an arbitrary  $h/L$  ratio [Fig. 4(e)], the horizontal coordinates of the response curve of  $\sigma$  versus  $u/L$  represent a weighted average of distances  $c$  and  $d$  of Fig. 4(a). As the ratio is varied, the response paths can occupy a continuous fan of directions as shown in Fig. 4(f). So there are infinitely many paths emanating from each bifurcation point, among which the extreme ones correspond to  $h = l$  and  $h = L$ , and there is a continuous sequence of such bifurcation points. Still, however, the states on all these postbifurcation paths can be stable. This is the case, e.g., for points 1 and 2 in Fig. 4(f).

The limits of stable states have recently been determined for flexural softening in beams or frames (Bažant et al. 1987). Bifurcation of the stable path may again occur earlier than these limits.

Multiplicity of response paths that consist entirely of stable states no doubt exists in inelastic finite element systems with damage or softening (Bažant 1976, 1977, 1985, 1986) but often goes undetected. An example was published (Bažant 1985, 1986) where various alternatives to the growth of a strain-softening (cracking) zone permitted two different equilibrium paths of load versus displacement. Step-by-step loading with iterations in each step produced as output only one of these paths (unless the system was forced by a suitable small disturbance to follow the other path). Yet the path that the finite element program produced as output turned out to be impossible in reality since another undetected path was subsequently found to minimize the increments of the Helmholtz free energy (or maximize the values of internal entropy). In spite of this fact, the computations which led to the incorrect path appeared to be stable, the iterations converged well.

Obviously, without actually checking for path stability, the inelastic finite element analysis as currently practiced can be deceptive. It does not necessarily yield the correct response path and may miss the stable one. Further research on numerical implementation of the present criteria is needed.

## STRUCTURE WITH SINGLE LOAD

Determination of the stable path is simple when there is only one applied load,  $f$ , or one controlled displacement,  $q$ . Noting that the second-order work  $(1/2)\delta f \delta q$  is represented by the cross-hatched triangles in Figs. 4 or 6, or that  $\delta^2 W = (1/2)K\delta q^2$ ,  $\delta^2 \bar{W} = 1/2 \delta f^2/K$  where  $K = df/dq$ , we have the following theorem: If the initial state is stable, the stable path is that for which the slope  $K$ , i.e., the tangential stiffness is minimum.

Examples proving this simple theorem are shown in Fig. 6. The solid arrows (with the triangles of  $\delta^2 W$  or  $\delta^2 \bar{W}$ ) denote the stable paths, and the

dashed arrows the unstable paths. For the states 3, 4, 7 no path is stable because these states are unstable (as revealed by snapback).

**IDENTIFICATION OF SOFTENING STRESS-STRAIN RELATIONS FROM TESTS**

The strain localization due to strain-softening complicates experimental evaluation of constitutive relations. Previous works have generally assumed the strain in small test specimens to remain uniform even after the peak stress. However, this can be assumed only for small specimens whose size is roughly equal to the characteristic length  $l$  of the material.

To simplify the material identification and avoid complicated finite element analysis of the test specimen, one may assume the post-peak specimen behavior to be approximately equivalent to a series coupling of a loading zone of volume fraction  $f$  which undergoes uniform strain-softening and an unloading zone of volume fraction  $1 - f$ , which undergoes unloading. The observed (measured) mean strain is  $\bar{\epsilon} = f\epsilon + (1 - f)\epsilon_u$  where  $\epsilon =$  true post-peak strain in the loading (softening) zone while  $\epsilon_u =$  strain in the unloading

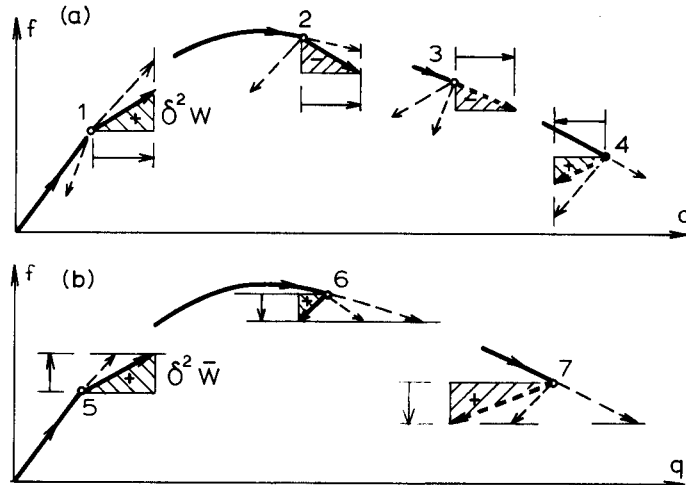


FIG. 6. Stable Post-Bifurcation Paths for Structures with Single Load  $P$ ; (a) for Displacement Control, (b) for Load Control

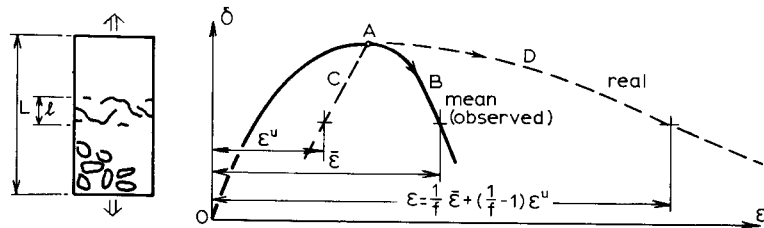


FIG. 7. Simple Approximate Estimation of Real Stress-Strain Relation from Observed Average Stress-Strain Relation

zone, which follows the unloading branch from the peak stress point (Fig. 7). From this,

$$\epsilon = \frac{1}{f} \bar{\epsilon} + \left( \frac{1}{f} - 1 \right) \epsilon^u \dots \dots \dots (35)$$

For tensile specimens, one may assume that  $f \approx l/L$ ; however, for specimens that exhibit post-peak softening in compression, shear or other complex modes, the series coupling assumption is questionable and the choice of  $f$  is more problematic (since its value might be substantially larger than  $l/L$ ). It should be noted that ideas similar to Eq. 35 have been advanced by M. Ortiz (seminar at Northwestern University on February 26, 1988).

**CONCLUSIONS**

1. An equilibrium state of an inelastic structure is stable if no deviation from this state can lead to an increase of internal entropy of the structure. It follows that an equilibrium state is stable if the second-order work is positive for every kinematically admissible deviation from the equilibrium state, and unstable if it is negative for some deviation.
2. While stability of a state is decided by considering deviations away from equilibrium, stability of a path is decided by considering approaches toward equilibrium.
3. Among all the equilibrium paths emanating from a bifurcation point, the internal entropy of the structure is maximized for the stable path. It follows that, among all the equilibrium paths, the stable path is that which consists of stable states and either minimizes the second-order work along the path if the displacements are controlled, or maximizes it if the loads are controlled, as compared to all other equilibrium paths.
4. The undeflected states of Shanley's perfect elastoplastic column are stable up to the reduced modulus load  $P_r$ , if the axial load is controlled, and up to an even higher load if the displacements are controlled. However, the stable undeflected states for loads  $P$  above the tangent modulus load  $P_t$  are not reachable in a continuous loading process, except when  $E_t$  decreases discontinuously. The stable equilibrium path is such that the deflection becomes nonzero as soon as  $P$  exceeds  $P_r$ .
5. If there is only a single load, the stable path is that for which the tangential stiffness is minimum, provided the initial state is stable.
6. Strain localization in strain-softening uniaxially stressed specimens must begin right after the peak stress state.

**ACKNOWLEDGMENT**

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