

SOLIDIFICATION THEORY FOR AGING CREEP

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ABSTRACT

The paper presents a new theory for the basic creep of concrete which takes aging into account in a manner which is better justified physically, better agrees with test results, and is more efficient computationally. The aging is treated as a consequence of volume growth of the load-bearing solidified matter (hydrated cement) whose properties are nonaging and are described by a Kelvin chain with age-independent moduli and viscosities. The thermodynamic restrictions on the aging creep law are satisfied, and the condition of nondivergence of creep curves is met. There are only four free material parameters, and they can be identified from the given test data by linear regression. Deviations from the principle of superposition are also correctly modeled by the theory. Good agreement with typical test data on creep and creep recovery is demonstrated.

Introduction

The phenomenon of aging, which is caused by cement hydration and probably also by gradual formation of bonds akin to polymerization, causes major complications for the modeling of concrete creep. The moduli and viscosities of spring-dashpot rheologic models depend on time, which complicates not only structural creep analysis but also their identification from test data. Thermodynamic restrictions are difficult to formulate and are found to be partially violated for those models for which they can be checked. The existing models which describe well the short-time creep violate the condition of nondivergence, causing that nonmonotonic recovery may be obtained from the principle of superposition. Deviations from the principle of superposition such as the adaptation nonlinearity are difficult to describe with the existing models.

The intent of this paper is to circumvent these problems by advancing a new theory whose basic mathematical form was proposed in Ref. 1 and was briefly summarized at a recent conference (2). A much more detailed presentation of the theory is given in Ref. 3.

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Volume Fraction Growth as a Measure of Aging

The new theory rests on the idea that the aging aspect of concrete creep is due to growth of the volume fraction $v(t)$ of the load-bearing solidified matter (i.e. hydrated cement), the properties of which are age-independent. Thermodynamic analysis is generally impossible for systems of substances whose properties vary with age. As known from chemical thermodynamics, time-dependence of any system's properties must be treated as a consequence of a time-varying composition of the system, which is in our case characterized by $v(t)$.

In the most simple form, it may be assumed that the volume, v , of hydrated cement grows by deposition of layers of solidified matter as shown in Fig. 1 (4). Let $\sigma_g(v, t)$ be the stress at time t in the layer which solidified when the total volume of the solidified matter was v . Now an essential point is that, at the moment it solidifies, the layer (dv) must be stress-free, i.e. $\sigma_g[v(\tau), \tau] = 0$. It follows that the nonaging viscoelastic stress-strain relation for the layer which solidified at time τ is (4)

$$\epsilon^v(t) - \epsilon^v(\tau) = \int_{\tau}^t \phi(t - t') \sigma_g[v(\tau), dt'] \quad (1)$$

in which $\sigma_g[v(\tau), dt'] = 0$ for $t' < \tau$; ϵ^v = viscoelastic strain due to solidified matter (hydrated cement) and $\phi(t - t')$ = microscopic creep compliance function of the solidified matter, representing the strain at age t caused by a unit microstress applied at age τ .

Since $v(t)$ is the only variable which introduces aging, a discrepancy might seem to exist due to the fact that the change of creep curves with age t' is known to be strong up to very high ages exceeding 10 years while the volume increase of hydrated cement terminates at the age of about 1 month. However, one must realize that further bonds continue to form even in the hydrated cement, as evidenced by the phenomenon of polymerization of tricalcium silicate. What matters for our purpose is the effective load-bearing volume in which the solidified matter has enough bonds to be sufficiently stiff, while the matter in the remaining volume, which has few bonds and is soft, must be discounted. The salient property is that the new bonds can be assumed to be stress-free at the time they form, and so this phenomenon can be included in Eq. 1 corresponding to Fig. 1 (v = load-bearing part of volume).

Now an important point is that the layer $dv(t')$ must be stress-free at the moment it solidifies, i.e. $\sigma_g[v(\tau), \tau] = 0$. Using this fact, along with Eq. 1 and the condition of equilibrium with the macroscopic applied stress σ , $\int_0^t \sigma_g[v(\tau), t] dv(\tau) = \sigma(t)$, Bazant (4) showed that σ_g can be eliminated from these equations, yielding a macroscopic stress-strain relation of the form:

$$\dot{\epsilon}^v(t) = \frac{F[\sigma(t)]}{v(t)} \dot{\gamma}(t), \quad \dot{\gamma}(t) = \int_0^t \dot{\phi}(t - t') d\sigma(t') \quad (2)$$

in which $\dot{\phi}(t - t') = \partial \phi(t - t') / \partial t$. A generalization for nonlinear behavior is introduced by inserting function $F(\sigma)$. $\gamma(t)$ can be regarded as the viscoelastic microstrain.

The assumption that the material must solidify in a stress-free state, $\sigma_g[v(\tau), \tau] = 0$, is applicable only to solidification at a solid-solution interface, as shown in Fig. 1. Conceivably, the solidification process could also take place at a solid-solid interface, in which case we could have a pressure across the interface, known as the crystal growth pressure. Consideration of such phenomena, however, is not germane to the age-dependence of creep. Anyhow it would require a model that is more complex than the simple parallel coupling of elements $dv(t')$ in Fig. 1.

AGING, CREEP, SOLIDIFICATION THEORY

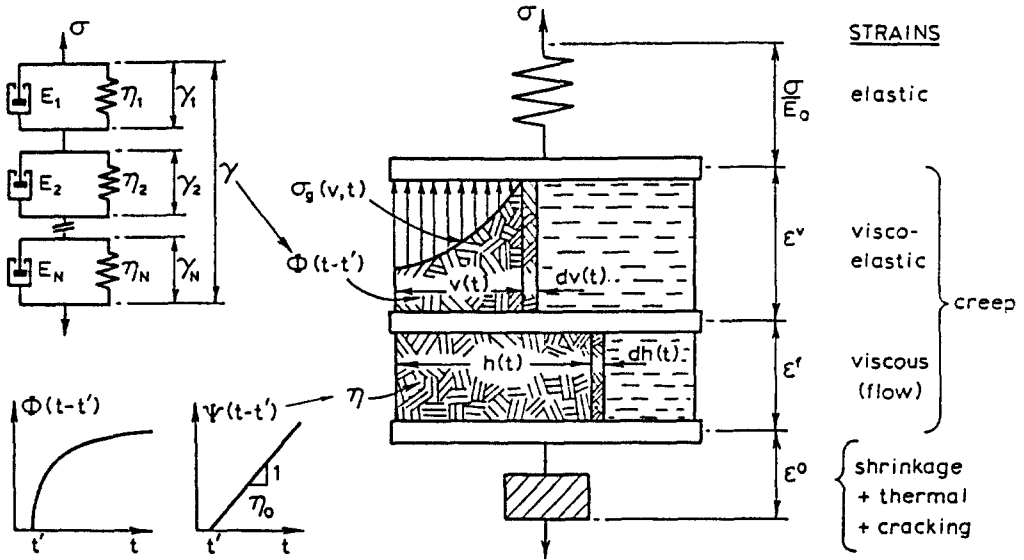


Fig. 1 - Model for the Role of Solidification in Creep

Values of $Q(t, t')$

| log t' | log (t - t') | | | | | | | | |
|--------|--------------|---------|---------|---------|---------|--------|--------|--------|--------|
| | -2 | -1 | 0 | 1 | 2 | 3 | 4 | 5 | ∞ |
| 0 | .4890 | .5826 | .6754 | .7352 | .7597 | .7684 | .7714 | .7724 | .7729 |
| 1 | .1547 | .1848 | .2185 | .2514 | .2724 | .2808 | .2838 | .2848 | .2853 |
| 2 | .04892 | .05846 | .06929 | .08123 | .09276 | .1000 | .1029 | .1039 | .1044 |
| 3 | .01547 | .01849 | .02192 | .02576 | .02994 | .03393 | .03641 | .03739 | .03789 |
| 4 | .004892 | .005846 | .006931 | .008149 | .009494 | .01094 | .01230 | .01314 | .01363 |

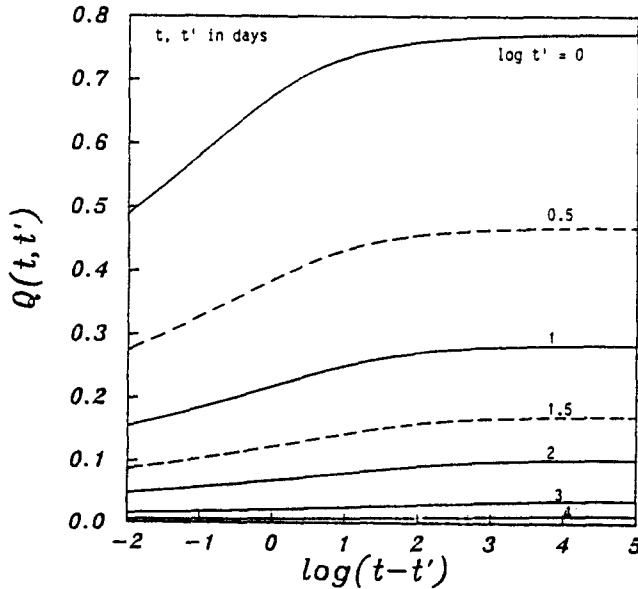


Fig. 2 - Function that Characterizes the Aging Viscoelastic Strain
 t = current time, t' = age at loading, in days.

Analysis of test data has indicated that, in addition to ϵ^v , concrete creep includes another component, ϵ^f , called flow, which is also affected by aging but is purely viscous (Fig. 1) rather than generally viscoelastic. It is described by an equation similar to Eq. 2 in which $\phi(t-t')$ is replaced by $\Psi(t-t') = (t-t')/\eta_0$ where η_0 = effective viscosity of the hydrated cement; therefore $\int_0^t \dot{\Psi}(t-t')d\sigma(t') = \sigma(t)/\eta_0$ and, in analogy to Eq. 2, we have

$$\dot{\epsilon}^f(t) = \frac{F[\sigma(t)]}{v(t)} \sigma(t). \tag{3}$$

Constitutive Relation for Creep

The total strain of concrete may be expressed as

$$\epsilon = \frac{\sigma}{E_0} + \epsilon^c + \epsilon^0, \quad \epsilon^c = \epsilon^v + \epsilon^f \tag{4}$$

where ϵ^f = total creep strain, ϵ^0 = shrinkage or thermal expansion, and E_0 = instantaneous elastic modulus. Similarly to the previously justified double power law (5,6,7,8), modulus E_0 is considered to represent the asymptotic elastic modulus for extremely fast (instantaneous) loading. This definition makes it possible to consider E_0 to be constant. The conventional static modulus, which depends on age, is then obtained as the inverse of the compliance function value for loading duration $t-t' \approx 0.001$ to 0.1 day, which includes the rapid initial creep. One might question that E_0 is considered to be independent of the age, t' . However, as justified previously for the double power law (5,6,7,8), the effect of age on $E(t)$ seems to be adequately included in the rapid initial creep contribution to the conventional elastic modulus.

The empirical functions in Eqs. 2-3 are introduced in the form

$$\phi(t-t') = q_2 \ln(1 + \xi^n), \quad \xi = (t-t')/\lambda_0 \tag{5}$$

$$\eta(t)^{-1} = q_4 t^{-1} \tag{6}$$

$$v(t)^{-1} = (\lambda_0/t)^m + \alpha \tag{7}$$

$$F[\sigma(t)] = \frac{1 + s^2}{1 - \omega}, \quad s = \frac{\sigma(t)}{f_c} \tag{8}$$

where $q_2, q_4, \alpha, n, m, \lambda_0$ = empirical constants, and $\omega = s^{10}$ = damage, which is negligible for $s < 0.7$. Note that Eq. 6 implies $\Psi(t-t') = q_4 \ln(t/t')$.

Expressing the total strain rate for $\epsilon^0 = 0$ according to Eqs. 2-4, and integrating, one finds that for a constant stress σ applied at age t' ,

$$\frac{\epsilon(t)}{\sigma} = J(t,t',\sigma) = q_1 + q_2 F(\sigma) Q(t,t') + q_3 F(\sigma) \ln[1 + (\frac{t-t'}{\lambda_0})^n] + q_4 \ln \frac{t}{t'} \tag{9}$$

in which q_1, \dots, q_4 = empirical constants, $q_1 = 1/E_0$, $q_3 = \alpha q_2$, and

$$Q(t,t') = \int_{t'}^t \left(\frac{\lambda_0}{\tau}\right)^m \frac{n(\tau-t')^{n-1}}{\lambda_0^n + (\tau-t')^n} d\tau. \tag{10}$$

$J(t,t',\sigma)$ is the secant compliance function at constant stress σ , and the functions multiplying q_2, q_3 and q_4 represent the nondimensionalized forms of the aging viscoelastic compliance, the nonaging viscoelastic compliance and the viscous (flow) compliance, respectively.

Experience with data fitting indicated that three material constants may

be fixed for all concretes:

$$n = 0.1, \quad m = 0.5, \quad \lambda_0 = 1 \text{ day.} \quad (11)$$

Thus there remain only four unknown material parameters, q_1 , q_2 , q_3 and q_4 , to be determined from the test data for a given concrete.

It is an advantage of the present formulation that the compliance involves all the unknown material parameters linearly. This makes it possible to determine all the unknown material parameters by linear regression.

The integral in Eq. 10 cannot be expressed in a closed form, but it can easily be evaluated and tabulated numerically; see Fig. 2. An approximate closed-form expression for $n = 0.1$, $m = 0.5$ and $\lambda_0 = 1$ day has been found:

$$Q(t, t') \sim Q_f \left[1 + \left(\frac{Q_f}{Z} \right)^r \right]^{-1/r} \quad (12)$$

$$\text{with} \quad Z = t'^{-m} \ln[1 + (t - t')^n] \quad (13)$$

in which

$$\left. \begin{aligned} \log Q_f &= -[0.1120 + 0.4308 \log t' + 0.0019 (\log t')^2], \\ r &= 1.7 t'^{0.12} + 8 \end{aligned} \right\} \quad (14)$$

see Fig. 2; t and t' must be in days and $\log = \log_{10}$, $\ln = \ln_e$. Q_f represents the final asymptotic values for $t = t' \rightarrow \infty$. The error of the formula for Q_f is within $\pm 0.09\%$ of Q_f and the coefficient of variation of errors is 0.01% . The error in Q is within $\pm 0.5\%$ of Q and the coefficient of variation of the errors is 0.2% .

It may be checked that for $t - t' \ll t'$, Eq. 9 approaches asymptotically the double power law. For $t - t' \gg t'$, the asymptotic form of Eq. 9 is a logarithmic law of the form $\epsilon = A_1 \ln t + A_2(t')$.

Another important advantageous property is that, according to Eq. 9, the condition

$$\frac{\partial^2 J(t, t', \sigma)}{\partial t \partial t'} \geq 0 \quad (15)$$

is always satisfied. This means that the creep curves for various ages at loading never diverge, according to Eq. 9. A further implication is that the creep recovery curves obtained by using the principle of superposition decrease always monotonically.

By virtue of introducing the nonlinearity in terms of the strain rate, rather than the strain, and describing the instantaneous strain as linearly elastic, the deviations from the principle of superposition accumulate with the load duration. This agrees with test data (9,10) and makes it possible to obtain a reduction of creep recovery compared to the prediction from the principle of superposition. Making function F dependent only on the current stress $\sigma(t')$ is a considerable simplification. However, the resulting nonlinearity of strain ϵ with respect to σ does depend on the past stress history, not just on the current stress.

It is interesting to note that the present formulation represents a compromise between the double power law (7,8) and the improved Dischinger model

used by CEB (11). The double power law, which describes well the short-time creep of concrete loaded at a young age and also both the short-time and long-time creeps of concrete loaded at an old age, is the limit case of the terms with q_2 and q_3 . The Dischinger model, which describes well the long-time creep of concrete loaded at a young age, is characterized by a flow term of the type $\phi(t) - \phi(t')$, which is here identical to the term with q_4 if one sets $\phi(t) = \ln t$. The term with q_3 is similar to the delayed elastic term in the improved Dischinger model, due to the fact that it is a nonaging function of $(t - t')$; however, unlike delayed elasticity, the term with q_3 has no final asymptotic value. The term with q_2 is lacking from the improved Dischinger models, but it is found that this term cannot be omitted if the typical test results should be matched closely.

Rate-Type Approximation and Numerical Integration

The main practical advantage of the present formulation is that it can be reduced to a rate-type creep law based on a rheologic model with nonaging properties. To obtain this formulation, the viscoelastic microstrain $\gamma(t)$ (Eq. 2) may be represented by a Kelvin chain (Fig. 1) with age-independent elastic moduli E_μ and viscosities η_μ . This leads to the relations:

$$\eta_\mu \dot{\gamma}_\mu + E_\mu \gamma_\mu = \sigma, \quad \gamma = \sum_{\mu=1}^N \gamma_\mu \quad (16)$$

which represent first-order linear differential equations for strain γ_μ of the μ -th Kelvin unit. An age-dependent Kelvin chain, by contrast, would lead to second-order differential equations for γ_μ (6,5,7). Integration of Eq. 16 for the case of constant stress σ applied at age t' yields:

$$\gamma(t) = \sigma \sum_{\mu=1}^N \frac{1}{E_\mu} \left(1 - e^{-(t-t')/\tau_\mu} \right), \quad \tau_\mu = \frac{\eta_\mu}{E_\mu} \quad (17)$$

where τ_μ are called the retardation times. For constant stress σ , the present model yields $\gamma(t) = \sigma \phi(t-t') = \sigma q_2 \ln(1 + \xi^n)$. Therefore, the following approximation is required:

$$\ln(1 + \xi^n) \approx \sum_{\mu=1}^N A_\mu \left(1 - e^{-\xi/\tau_\mu} \right). \quad (18)$$

If A_μ is determined, then $E_\mu = 1/(q_2 A_\mu)$. It appears that a rather accurate approximation of $\ln(1 + \xi^n)$ within the range $\tau_2 \leq \xi \leq 0.1 \tau_N$ is possible with the choice $\tau_\mu = 10^{\mu-2} \tau_2$ for $\mu \geq 2$ and $\tau_1 = 10^{-5} \tau_2$. Coefficients A_μ for this approximation may be obtained by the method of least squares. However, an explicit approximate formula has been found:

$$A_\mu = b_\mu \tau_\mu^{m(\mu)} \quad (19)$$

in which $m(\mu) = n/(1 + c^z \mu^z)$, $c = 0.146 n^{-0.1}$, $b_\mu = 1.1 n(1-n^3)$ for $\mu = 2, \dots, N-1$, $b_N = 1.5 n^{1.25}$, and b_1, z are given by Table 1. The error of the approximation in Eq. 17 for $m = 0.5$ and $n = 0.1$ has, within the range $\tau_2 \leq \xi \leq 0.1 \tau_N$, the maximum of only 0.7%. Within the range $0.25 \tau_2 \leq \xi \leq 0.25 \tau_N$, the maximum error is 1.15%, and the coefficient of variation of the errors is 0.46%.

Owing to the fact that the foregoing method always yields nonnegative E_μ and η_μ , the thermodynamic restrictions are satisfied. In similarity to previous rate-type models, the effect of temperature on the creep rate can be introduced in Eq. 15 by replacing $\eta_\mu = E_\mu \tau_\mu$ with $E_\mu \tau_\mu f(T)$ where $f(T)$ depends on temperature according to the activation energy theory. Furthermore, variation of temperature requires that $v(t)$ be replaced with $v(t_e)$ where t_e is the

equivalent hydration period (5-7).

To permit structural creep analysis with increasing time steps Δt which can become much larger than the shortest retardation time, an exponential algorithm must be used (12,7,5). The incremental stress-strain relations are obtained by integrating Eq. 15 exactly under the assumption that $\sigma(t)$ varies linearly from t_i to t_{i+1} . This yields:

$$\gamma_{\mu, i+1} = \gamma_{\mu, i} e^{-\Delta y_{\mu}} + \frac{\sigma_i}{E_{\mu}} (1 - e^{-\Delta y_{\mu}}) + \frac{1 - \lambda_{\mu}}{E_{\mu}} \Delta \sigma \quad (20)$$

where $\Delta y_{\mu} = \Delta t / \tau_{\mu}$ and $\lambda_{\mu} = (1 - e^{-\Delta y_{\mu}}) / \Delta y_{\mu}$, and subscripts i and $i+1$ refer to times t_i and t_{i+1} . Eq. 19 leads to the stress-strain relation $\Delta \gamma = (\Delta \sigma / D) + \Delta \gamma''$ in which

$$\frac{1}{D} = \sum_{\mu=1}^N \frac{1 - \lambda_{\mu}}{E_{\mu}}, \quad \Delta \gamma'' = \sum_{\mu=1}^N \left(\frac{\sigma_i}{E_{\mu}} - \gamma_{\mu, i} \right) (1 - e^{-\Delta y_{\mu}}) \quad (21)$$

and $\Delta y_{\mu} = \Delta t / \tau_{\mu}$, $\lambda_{\mu} = (1 - e^{-\Delta y_{\mu}}) / \Delta y_{\mu}$; subscript i refers to time t_i . It may be noted that Eqs. 20 - 21 are a special case of the equations of the exponential algorithm for an aging Kelvin chain (12) and are equivalent to those presented for a nonaging Kelvin chain by Taylor, Pister and Goudreau (13), and by Zienkiewicz and Watson (14); but an important difference is that here these equations apply to the microstrain $\gamma(t)$ rather than the total strain $\epsilon(t)$. For $\epsilon(t)$, one obtains the quasielastic stress-strain relation

$$\Delta \sigma = E'' (\Delta \epsilon - \Delta \epsilon'') \quad (22)$$

in which

$$\frac{1}{E''} = q_1 + \frac{1}{D} \frac{F(\sigma_{i+\frac{1}{2}})}{v_{i+\frac{1}{2}}} \quad (23)$$

$$\Delta \epsilon'' = F(\sigma_{i+\frac{1}{2}}) \left(\frac{\Delta \gamma''}{v_{i+\frac{1}{2}}} + \frac{q_2}{t_{i+\frac{1}{2}}} \Delta t \right) + \Delta \epsilon^0 \quad (24)$$

Subscript $i + \frac{1}{2}$ refers to time $t_{i+\frac{1}{2}} = t_0 + [(t_i - t_0)(t_{i+1} - t_0)]^{\frac{1}{2}}$, where $t_0 =$ time of first loading.

Eq. 22 reduces the solution of any creep problem to a sequence of elastic solutions with initial strains. Due to nonlinearity, iterations of each time step are needed to achieve good accuracy.

Verification by Test Data

The present formulation has been compared in Ref. 3 to numerous test data from the literature. Some of the typical comparisons are shown in Fig. 3 and Fig. 4. The fit of these data is certainly satisfactory, both for the tests at constant stress (Fig. 3) and at step-wise stress histories (Fig. 4). The dashed lines in Fig. 4 represent predictions according to the principle of superposition, which are obviously worse. For further comparisons with test data, see Ref. 3.

Conclusions

1. The solidification theory for basic creep makes it possible to avoid using an aging Maxwell or Kelvin chain model in creep structural analysis. Rather, the analysis can be based on a Kelvin chain which has age-independent elastic moduli and viscosities and describes creep microstrain.
2. The description of creep with elastic deformation involves only four

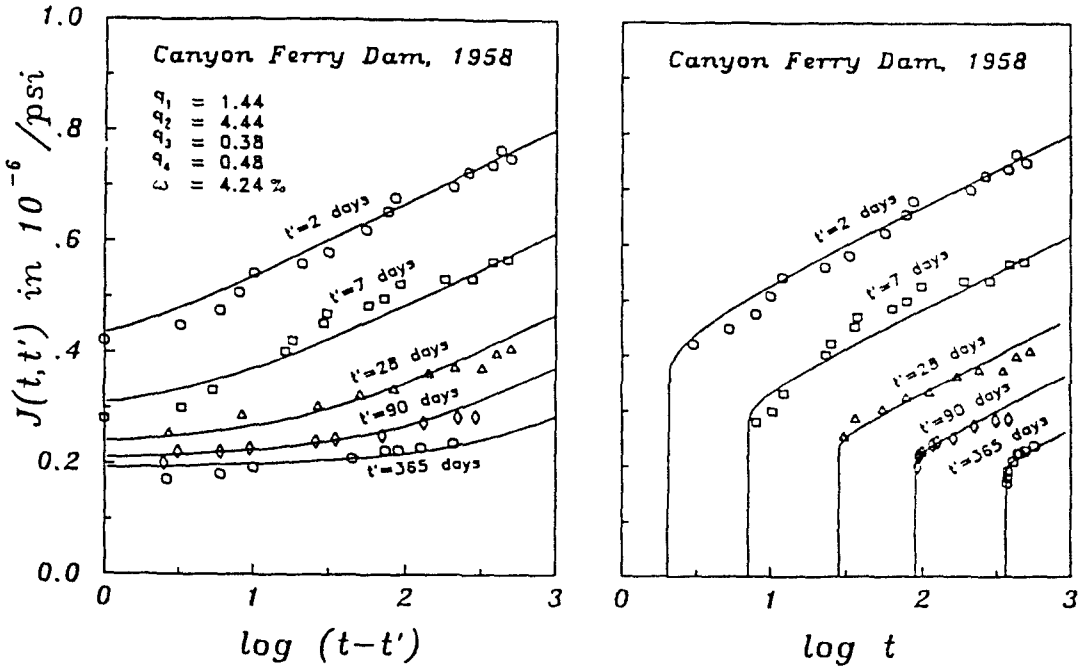


Fig. 3 - Comparison with Laboratory Test Data for Canyon Ferry Dam Concrete (Hanson, 1953; Hanson and Harboe, 1958).

free material parameters which can be determined from the given test data by linear regression.

3. The creep curves for different ages at loading never exhibit divergence. The predicted creep recovery is always monotonic. The fits of test data are excellent, and better than with previous models.

4. Deviations from the principle of superposition in the service stress range are represented correctly, in agreement with test data.

5. The model is justified by simplified micromechanics analysis of the solidification process of portland cement, in which aging is obtained as a consequence of the growth of volume of a nonaging solid (its load-bearing part).

6. For short load durations, the model is asymptotically equivalent to the double power law, and for long load durations to the logarithmic law.

Acknowledgment

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Table 1. - Optimum Parameters for Dirichlet Series Expansion

| | | | | | | | | | | | | | | | | | | |
|----------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| n | 0.01 | 0.03 | 0.05 | 0.07 | 0.09 | 0.11 | 0.13 | 0.15 | 0.17 | 0.19 | 0.21 | 0.23 | 0.25 | 0.27 | 0.29 | 0.31 | 0.33 | 0.35 |
| b ₁ | 0.674 | 0.632 | 0.587 | 0.538 | 0.491 | 0.507 | 0.529 | 0.552 | 0.574 | 0.593 | 0.610 | 0.624 | 0.634 | 0.641 | 0.643 | 0.641 | 0.635 | 0.624 |
| b ₂ | 0 | 0 | 0 | 0 | 0.039 | 0.355 | 0.621 | 0.860 | 1.080 | 1.288 | 1.486 | 1.677 | 1.860 | 2.037 | 2.207 | 2.372 | 2.531 | 2.683 |

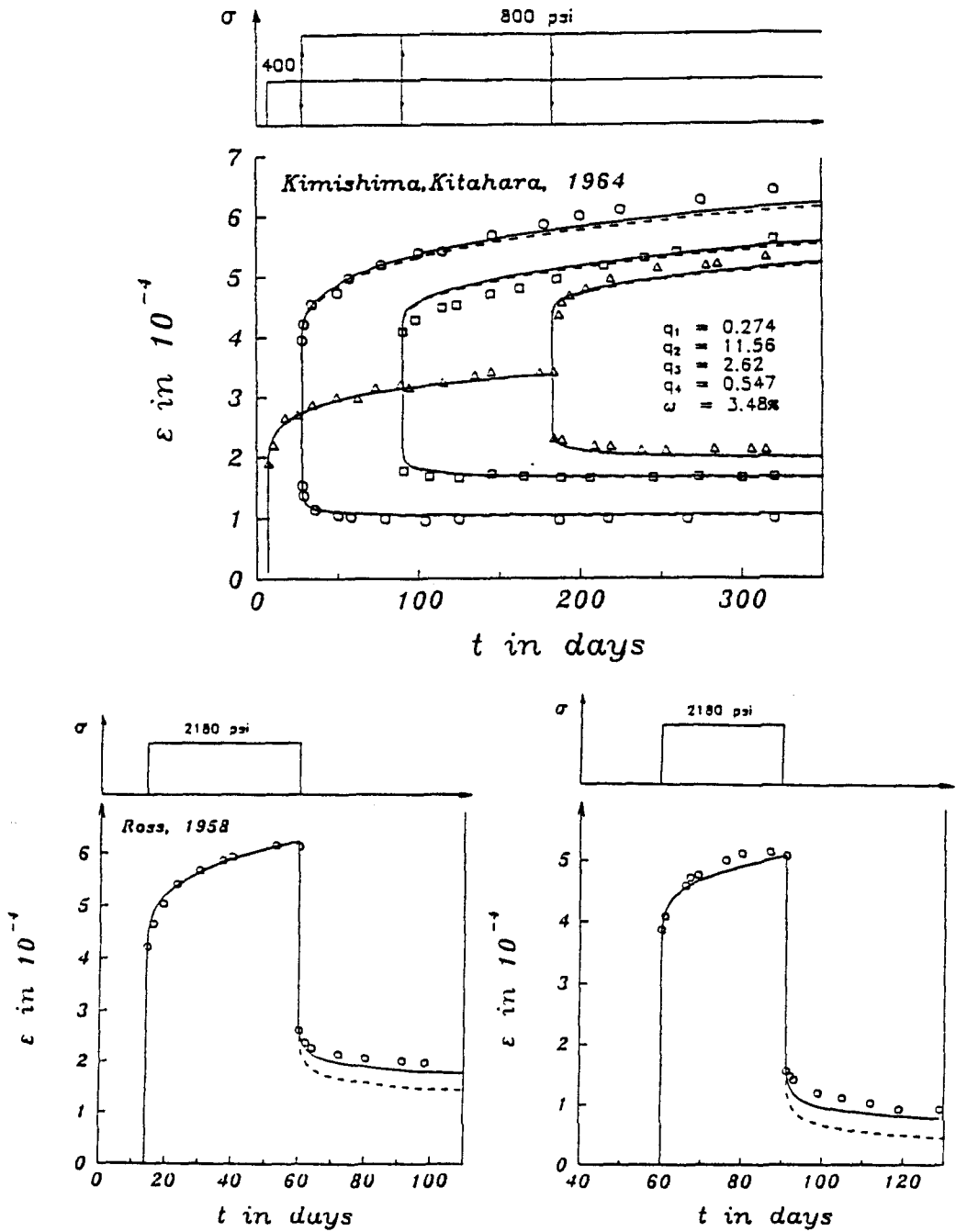


Fig. 4 - Comparison with Test Data of Kimishima and Kitahara (1964), and Ross (1958).

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Appendix - Parameter Prediction

When no test data are available for the given concrete, parameters q_1 , q_2 , q_3 , q_4 may be approximately predicted from the following formulas obtained by statistical analysis of numerous test data from the literature:

$$q_1 = 12.5 (w/c)^{3.5} \quad (25)$$

$$q_2 = -22.8 + 2.5 \ln [(w/c)^5 (a/c) f'_c{}^{1.5}] \quad (26)$$

$$q_3 = 16000 [(w/c)^4 (a/c) f'_c{}^{0.4}]^{-0.8} \quad (27)$$

$$q_4 = 0.000082 (w/c) (a/s) f'_c \quad (28)$$

in which q_1, \dots, q_4 are in psi^{-1} , $f'_c = 28$ day cylindrical compression strength in psi ($1 \text{ psi} = 6895 \text{ Pa}$), w/c = water cement ratio of the mix, a/c = aggregate cement ratio, s/c = sand-cement ratio (all by weight), and sand is defined as the aggregate less than 4.7mm in size (sieve No. 4).