# Crack Spacing in Reinforced Concrete: Approximate Solution

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In a companion paper in this issue (1), a simplified theory for predicting the spacing of cracks in a parallel crack system in reinforced concrete was presented. Due to the approximate nature of this theory, it would be reassuring to show that a different approximate analysis yields about the same result. For this purpose, we will now show a solution based on the formulas for the stress caused by a concentrated force in a half space. [All notations are the same as in the companion paper (1) and are summarized in Appendix III of this paper. The reader should also refer to this paper for an explanation of the type of fracture mechanics energy approach used.]

We now consider only sparse full length cracks, such that $s > b/2$, because they are more likely to happen and are also simpler to analyze. We suppose that the total accumulated bond force, $P_b$, is transmitted from the steel bar into concrete at one point at a distance $\beta s/2$ from the crack, in which $\beta$ = an empirical coefficient between 0.0 and 2.0, probably $\beta \simeq 1.0$, and we imagine that the volume of concrete which is stressed by this force is a hemisphere whose radius, $R$, is such that its volume $2\pi R^3/3$ is equal to the actual volume of concrete limited by a normal plane through the point of application of $P_b$, i.e., to $b_1 b_2(2s - \beta s)/2$ [Figs. 1(*a*)–(*b*)]. This yields

$$R = \left[\frac{3}{4\pi} As(2-\beta)\right]^{1/3}; \quad A = b_1 b_2 \qquad (1)$$

For the stress in the hemisphere we use the Boussinesq-Kelvin solution for $\nu = 1/2$ (Ref. 15 of preceding paper). This is the simplest form solution for the spread of a concentrated force. Although actual $\nu$ is about 0.2, we use $\nu = 0.5$ because the results are not very different and larger errors are caused by our other simplifications. For this solution, the stress state everywhere is a uniaxial stress in the radial direction, i.e.:

$$\sigma_{rr} = C_b \frac{\cos\theta}{r^2}; \quad C_b = \frac{3P_b}{2\pi} = \frac{3F_b s}{2\pi} \qquad (2)$$

All other stress components are zero. Here, $r$, $\theta$ = the spherical coordinates [Figs. 1(*a*)–(*b*)]; and $F_b$ = average distributed bond force.

We need to assure compatibility of the deformation due to $\sigma_{rr}$ with the average strain, $\epsilon_b$, of concrete at the bar surface due to $\sigma_{rr}$. This condition may be written as $\int_{D/2}^{z'_b} (\sigma_{rr})_{\theta=0}\, dr/E_c \simeq \epsilon_b s$, in which, for the sake of simplicity, we integrate along $\theta = 0$ rather than along the bar surface, in which case we must exclude $\theta \in (0, D/2)$ from the integration due to the displacement singularity of Boussinesq-Kelvin solution at $r = 0$ (for integration from $\theta = 0$ this integral would be infinite):

$$\epsilon_b = \frac{3}{\pi}\frac{F_b}{E_c}\left[\frac{1}{D} - \frac{1}{(2-\beta)s}\right] \qquad (3)$$

We now introduce an equivalent uniform strain of concrete, $\epsilon_c$, such that the strain energy due to $\epsilon_b$ is the same as for our nonuniform stress field, i.e., we require that

$$\frac{E_c}{2}\epsilon_c^2 b_1 b_2 s = \int_{r=D/2}^{R}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} \frac{1}{2E_c}\left(C_b \frac{\cos\theta}{r^2}\right)^2 r^2 \sin\theta\, d\phi\, d\theta\, dr \qquad (4)$$

in which $\theta$ = circumferential spherical coordinate. From this condition we get

$$\epsilon_c = \frac{C_b}{E_c}\left[\frac{2\pi}{3As}\left(\frac{2}{D} - \frac{1}{R}\right)\right]^{1/2} \qquad (5)$$

Now we consider the energy balance during crack formation, distinguishing the cases of bond slip and no slip.

The strain energy may be expressed as

$$U = \frac{E_s\epsilon_s^2}{2}\frac{\pi D^2}{4} s + \frac{4}{2E_c}\int_{r=D/2}^{R}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{\pi/2}\left(C_b \frac{\cos\theta}{r^2}\right)^2 r^2 \sin\theta\, d\phi\, d\theta\, dr \qquad (6)$$

or $$U(s) = \frac{\pi D^2}{8} E_s s \epsilon_s^2 + \frac{\pi C_b^2}{3E_c}\left(\frac{2}{D} - \frac{1}{R}\right) \qquad (7)$$

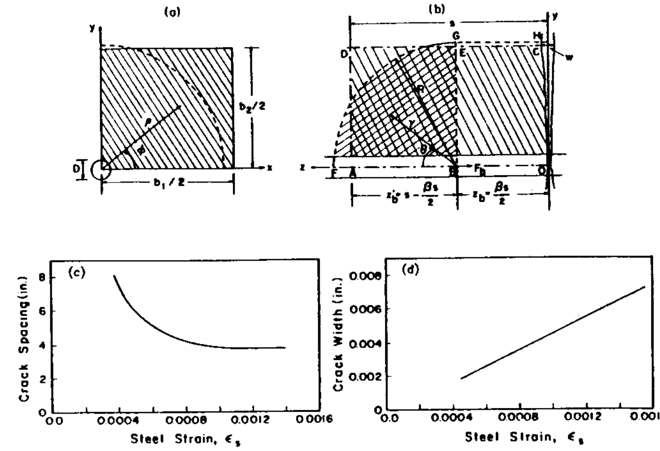


**FIG. 1.—(*a*, *b*) Cross Section of Cracked Reinforced Concrete; (*c*, *d*) Typical Diagrams for Concentrated Force Solution**

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for the strain energy release during fracture: (1) If there are no cracks in the initial state, we have

$$\Delta U = \frac{s}{2}\epsilon_s^2\left(\frac{\pi}{4}E_sD^2 + AE_c\right) - U(s) \quad (8)$$

(2) if there are cracks of spacing, $2s$, in the initial state, we have

$$\Delta U = U(2s) - U(s) \quad (9)$$

The energy consumed by crack formation is given by Eq. 16 of the preceding paper (1). When there is bond slip ($F_b = F_b'$, $\epsilon_b \le \epsilon_s$), further energy is consumed by the change, $\Delta u_b$, of bond slip displacements during the fracture formation, and, similarly to Eq. 18 of the preceding paper

$$\Delta W_b = F_b'\epsilon_b\frac{s^2}{2} \quad (10)$$

In the absence of bond slip ($F_b < F_b'$, $\epsilon_b = \epsilon_s$) we have $\Delta W_b \simeq 0$.

The crack width may be approximated:

At bar surface: $w = s(\epsilon_s - \epsilon_b)$ (11)

Away from bar: $w = s(\epsilon_s - \epsilon_c)$ (12)

The calculation procedure for this method is as follows:

**First Cracks:**

1. Assuming no bond slip, start calculation by solving Eqs. 3, 8, and Eq. 16 of the preceding paper for $s$ and $F_b$.
2. Substitute Eqs. 1–2 into Eq. 7, and then express $F_b$ from Eq. 3 and substitute it into Eq. 7, and then solve the equation $\Delta U = \Delta W_f$ (Eq. 8 = Eq. 16 of the preceding paper) for $s$, $\epsilon_s$ being specified.
3. Finally, check whether $F_b \le F_b'$. If not, set $F_b = F_b'$ and repeat the solution assuming the bond slip case (Eq. 10). The unknowns are then $s$ and $\epsilon_b$, and the procedure is the same except that Eq. 17 of Ref. 1 is used in step 2 instead of Eq. 16 of Ref. 1.

**Subsequent Cracks.**—Crack spacing, $s$, is now known, since it is one-half of the previous spacing, and we seek the strain, $\epsilon_s$, at which the spacing is halved:

1. If $F_b < F_b'$ for the previous spacing, assume no bond slip. The unknowns are then $\epsilon_s$ and $F_b$.
2. Substitute Eq. 3 for $F_b$ into Eq. 7, and then solve the equation $\Delta U = \Delta W_f$ (Eq. 9 = Eq. 16 of the preceding paper).
3. Check whether $F_b \le F_b'$, and, if not, set $F_b = F_b'$ and repeat calculation assuming bond slip (Eq. 10). The unknowns are now $\epsilon_s$ and $\epsilon_b$, but one can eliminate $\epsilon_b$ from Eq. 3. The procedure is the same except that Eq. 17 of Ref. 1 is used instead of Eq. 16 of Ref. 1.

To illustrate our formulation, an example of the calculation of crack spacing and crack width is shown in Figs. 1(*c*)–(*d*). The parameters are $b_1 = 2.24$ in., $b_2 = 7.5$ in., $D = 0.875$ in., $E_s = 29 \times 10^6$ psi, $E_c = 3.98 \times$



$10^6$ psi, $F_b' = 2{,}170$ lb/in., $\mathcal{G}_f = 0.896$ lb/in., and $\beta = 1.0$. These values are similar to those used in the preceding paper. The results plotted in Figs. 1(*c*)–(*d*) are rather close to those in the preceding paper which in turn were found to agree reasonably well with existing test data.

It would be possible to extend this theory to also cover the closely spaced shorter cracks which do not extend across the full width, $b$.

## Appendix I.—Integral versus Incremental Fracture Energy Criterion

Consider a penny-shaped crack of radius $b$ in an infinite elastic solid subjected at infinity to a uniform traction, $\sigma$, normal to the crack plane [Fig. 2(*a*)]. The exact solution gives, for the total strain energy loss of the infinite solid due to the crack, the expression $\Delta U = 8\sigma^2b^3/3E'$, in which $E' = E/(1 - \nu^2)$; $\nu$ = Poisson ratio; and $E$ = Young's modulus (Refs. 28, 29 of preceding paper). The strain energy release rate then is $\mathcal{G} = (1/2\pi b)\partial\Delta U/\partial b = 4\sigma^2 b/\pi E'$. The usual energy criterion of fracture mechanics requires that $\mathcal{G} = \mathcal{G}_f$ if the crack should grow, $\mathcal{G}_f$ being the fracture energy per unit area of the crack. This yields for $\sigma$ the well-known critical value needed to make the crack grow:

$$\sigma_{cr} = \sqrt{\frac{\pi E'\mathcal{G}_f}{4b}} \quad (13)$$

On the other hand, if we assume a simultaneous crack formation and apply the energy criterion integrally for the whole crack, as we do here, we have $\Delta U = \pi b^2\mathcal{G}_f$, which yields for $\sigma$ the value

$$\bar{\sigma}_{cr} = \sqrt{\frac{3\pi E'\mathcal{G}_f}{8b}} \quad (14)$$

This value is 1.225 times larger than $\sigma_{cr}$. Similarly, if we made the same consideration for an infinite planar elastic solid with a line crack, loaded at infinity by a uniform stress normal to the crack, $\bar{\sigma}_{cr}$ would be 1.41 times larger than $\sigma_{cr}$.

These differences between the usual incremental criterion and our integral criterion are explicable in terms of the R-curves, i.e., the resistance curves which indicate that the apparent (measured) fracture energy in a heterogeneous material is not constant at the start of crack propagation from a notch of depth $b_0$ but increases from an initial value, $\mathcal{G}_f^0$, until it reaches a certain asymptotic value, $\mathcal{G}_f^\infty$ [Fig. 2(*c*)]. The value of fracture

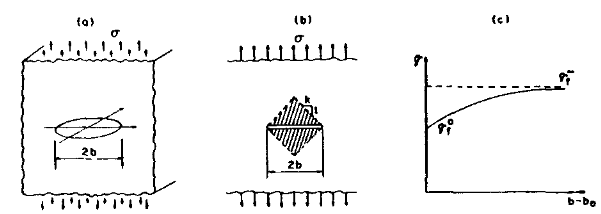


**FIG. 2.—(*a*, *b*) Crack in Infinite Elastic Solid; (*c*) Resistance Curve (R-Curve)**

energy which we should use in the incremental approach should be $\mathcal{G}_f^\infty$, and the one in our integral approach should be $\mathcal{G}_f^0$. Now, since $\mathcal{G}_f^0$ is normally 0.4–0.8 times $\mathcal{G}_f^\infty$, as known from tests, we see that replacing $\mathcal{G}_f^\infty$ with $\mathcal{G}_f^0$ in Eq. 14 can give a value, $\bar{\sigma}_{cr}$, which is consistent with $\sigma_{cr}$ from Eq. 13. This justifies our unorthodox energy approach to *small-scale* fracture of concrete.

To illustrate the approximate method of spreading of localized disturbances into an elastic medium according to inclined "stress lines," we may imagine that the strain energy relieved by the crack is the strain energy, $\Delta U$, originally contained in the cones whose cross section is cross-hatched in Fig. 2(*b*): $\Delta U = 2\pi b^2(b/3k)\sigma^2/2E$, in which $k$ = the empirical slope of the inclined stress line. Setting $\Delta U = \Delta W_f = \pi b^2 \mathcal{G}_f$, we obtain $\sigma = (3kE\mathcal{G}_f/b)^{1/2} = \bar{\sigma}_{cr}$. We see that this equation is of the same form as the exact result (Eq. 14), and the numerical factor is also the same when $k = \pi/8(1-\nu^2)$ or $k = 0.406$. If we would apply the usual incremental fracture criterion, we would set $\partial(\Delta U)/\partial b = 2\pi b \mathcal{G}_f$, which would yield $\sigma = (2kE\mathcal{G}_f/b)^{1/2} = \sigma_{cr}$. We see that $\bar{\sigma}_{cr}$ is now 1.225 times larger than $\sigma_{cr}$. (For the analogous planar problem of a line crack in an infinite plane, a similar illustration of the incremental fracture calculation by the method of inclined stress lines is given by Knott, Ref. 22 of preceding paper.)

## Appendix II.—Reference

1. Bažant, Z. P., and Oh, B. H., "Spacing of Cracks in Reinforced Concrete," *Journal of Structural Engineering*, ASCE, Vol. 109, No. 9, Sept., 1983, pp. 2066–2085.

## Appendix III.—Notation

*The following symbols are used in this paper:*

$A$ = cross-section area of concrete;
$b, b_1, b_2$ = spacings of bar;
$C_b$ = $3P_b/2\pi$;
$D$ = bar diameter;
$E_c, E_s$ = Young's moduli of concrete and steel bar, respectively;
$F_b'$ = ultimate bond force;
$f_c'$ = compressive strength of concrete;
$f_t'$ = tensile strength of concrete;
$\mathcal{G}_f$ = fracture energy;
$k$ = slope of stress line;
$P_b$ = accumulated (concentrated) bond force;
$R$ = radius of hemisphere;
$r$ = radial coordinate;
$s$ = crack spacing;
$U$ = strain energy;
$u_b$ = bond slip displacement;
$V$ = volume of concrete;
$W_b$ = work consumed by bond slip;
$W_f$ = work consumed by fracture;
$w$ = crack width;
$\beta$ = empirical coefficient;
$\epsilon_b, \epsilon_c$ = strains of concrete at and away from bar, respectively;
$\epsilon_s$ = strain of steel bar;
$\theta, \phi$ = spherical coordinates;
$\nu$ = Poisson's ratio; and
$\sigma_{rr}$ = radial stress of concrete.