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STABILITY OF PARALLEL CRACKS IN SOLIDS REINFORCED BY BARS

ZDENĚK P. BAŽANT† and ABU BAKR WAHAB‡
Department of Civil Engineering, Northwestern University, Evanston, IL 60201, U.S.A.

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Abstract—Investigated is the effect of one layer of steel reinforcement on the instability of a system of parallel equidistant shrinkage or cooling cracks in a concrete halfspace or parallel equidistant cracks due to bending of a beam. The cracks are assumed to propagate along straight lines normal to halfspace surface. The instability mode consists in the closing of every other crack at the expense of an extension and increase of the width of the remaining cracks. The previous formulation of stability conditions in terms of the derivatives of the stress intensity factors with respect to the crack lengths is used and numerical results are obtained by finite elements. It is found that instabilities of cracks in reinforced concrete do exist and are profoundly affected by the presence of reinforcement. Assuming a relatively limited length of bond slip near the crack, one finds that the presence of a reinforcement layer greatly increases the penetration depth of cooling or drying at which the instability occurs, but does not prevent the instability from occurring deeper beneath the reinforcement. A small amount of reinforcement, smaller than that required by the building codes, is sufficient to achieve this effect while a further increase of the reinforcement amount has relatively little effect.

1. NATURE OF PROBLEM

Reinforced concrete structures under service loads typically contain numerous cracks and their width has a profound effect on structural performance. The crack width must be kept small, generally less than 0.3–0.4 mm and preferably 0.1 mm [1], in order to assure that the rough crack surfaces would be interlocked and capable of transmitting shear stresses, that the fatigue and damage resulting from cracks would not be excessive, and that certain substances which can participate in reinforcement corrosion or other modes of deterioration of concrete (chloride ions, oxygen, water, carbon dioxide, sulphates, etc.) would not penetrate into concrete in significant amounts.

The average overall tensile strain of reinforced concrete approximately equals the sum of the widths of all cracks within length L, divided by L. Thus, the crack width is approximately in inverse proportion to the crack spacing. The problem of spacing of cracks in reinforced concrete is a classical one. One well known simple solution gives the spacing of cracks in the cover of reinforcing bars on the basis of the accumulated bond force and the tensile strength of the uncracked cover[2].

Recently, crack spacing has been investigated in connection with the cooling of hot rock for the purpose of geothermal heat extraction[3–6]. Realizing that in certain problems, such as the cooling cracks in a halfspace, fracture mechanics admits, for the same load, solutions of different crack spacings and lengths, stability of the crack system must be investigated. This was first done in [3] in which the conditions of stability of a system of cracks propagating in known directions were determined by analyzing the second variation of the work needed to create the cracks, as well as by formulating the conditions of adjacent equilibrium (a summary of this development was given earlier in Ref. [4]). The later work of some other authors on this problem was commented upon in [3]. The post-critical behavior was investigated in detail in [5] and the effect of cooling profile was analyzed in [6].

In these works it was found that instability of a system of parallel shrinkage or cooling cracks in a halfspace may cause some cracks to close and the remaining ones to extend and widen. This result applies not only for rock, but also for unreinforced concrete, and thus the question of the effect of the reinforcement on this instability naturally emerges. A study of this question is the purpose of this study.

The crack problem will be approached by linear fracture mechanics. This approach may often represent a crude approximation in case of concrete. However, we must resort to fracture

†Professor of Civil Engineering.
‡Postdoctoral Research Associate; on leave from the University of Khartoum, Khartoum, Sudan.

mechanics out of necessity, because no other consistent approach to the propagation of cracks in concrete is available at present[7]. The initiation of cracks in reinforced concrete has usually been treated by a simple concept of strength. However, this is inapplicable to crack propagation because the stress in the finite element just ahead of the crack due to the given load can be made arbitrarily large by choosing a sufficiently small element size[7]. A consistent and more realistic approach would be an analysis of the instability due to strain-softening and of strain localization in a nonlinear material; but this approach is not sufficiently developed at present.

2. REVIEW OF STABILITY CONDITIONS OF CRACK SYSTEM

Consider a brittle elastic solid (unreinforced or reinforced) containing a system of m cracks of lenth $a_i (i = 1, 2, \ldots, m)$ that propagate in Mode I along known paths. The crack system is considered to be stable (locally stable) if, for a specified loading and displacement boundary conditions, work must be supplied to obtain any admissible infinitesimal changes δa_i in the crack lengths. If this work, ΔW, can be negative for some admissible δa_i, the crack system is unstable. If this work is zero, we have a critical state, such that an adjacent equilibrium state of the crack system exists for the same loading. Thus, the solution is not unique in the critical state. If the crack length a_i is plotted as a function of some loading parameter (e.g. the penetration depth D of cooling or drying), the equilibrium path of the system exhibits a bifurcation at the critical state.

The admissible crack extensions δa_i are [3]:

$$\begin{aligned} &\text{for } K_i = K_{c_i}: && \delta a_i \geq 0; \\ &\text{for } 0 < K_i < K_{c_i}: && \delta a_i = 0; \\ &\text{for } K_i = 0: && \delta a_i \leq 0, \end{aligned} \tag{1}$$

where K_i = stress intensity factor of the ith crack, and K_{c_i} = critical stress intensity factor K_c (material property) of the ith crack; $K_i = \lim \sigma_y (2\pi r)^{1/2}$ for $r \to 0$ where r = distance from the crack tip and σ_y = normal stress on the crack extension line ahead of the tip[8]. As shown in [3], the stability of a crack system is decided by the sign of the second variation $\delta^2 W$ of work W needed to create the cracks;

$$\delta^2 W = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} W_{ij} \delta a_i \delta a_j \begin{cases} > \text{ stable (all admissible } \delta a_i) \\ = 0 \text{ critical (some admissible } \delta a_i) \\ < 0 \text{ unstable (some admissible } \delta a_i) \end{cases} \tag{2}$$

in which

$$W_{ij} = \frac{\partial^2 W}{\partial a_i \partial a_j} = \begin{cases} = K_i \dfrac{\partial K_i}{\partial a_j} + 2 \dfrac{\partial \gamma_i}{\partial a_i} & \text{for } i = j \text{ and } \delta a_i > 0 \quad \text{(3a)} \\ = K_i \dfrac{\partial K_i}{\partial a_j} & \text{for all other cases.} \quad \text{(3b)} \end{cases}$$

Here $\gamma_i = K_{c_i}^2 / 2E'$ = effective surface energy for the creation of crack surfaces[8]; $E' = E$ for plane stress and $E' = E/(1 - \nu^2)$ for plane strain, E = Young's modulus, and ν = Poisson ratio. If the material properties are uniform, $\partial \gamma_i / \partial a_i = 0$ (homogeneous solid). From the relation of K_i to the release rate of elastic energy it can be shown[3] that

$$K_i \frac{\partial K_i}{\partial a_j} = K_j \frac{\partial K_j}{\partial a_i} \tag{4}$$

which means that $W_{ij} = W_{ji}$.

3. PARALLEL CRACKS IN REINFORCED HALFSPACE

We will restrict attention to a system of parallel equidistant cracks in a reinforced homogeneous isotropic halfspace undergoing small strain. The cracks are considered orthogonal to the surface and are initially of equal length, $a_i = a_1$. Steel reinforcing bars which are uniformly and densely distributed in a plane parallel to the surface ($x = c = \text{const.}$) and are normal to the cracks are considered to be embedded in the solid. The situation is typical of reinforced concrete.

The loading of the halfspace is due to either cooling or shrinkage, such that the profile of the temperature change T or of the free unrestrained shrinkage strain ϵ_{sh} is the same along each normal to the halfspace surface. We assume that the cracks produced by this loading propagate only along straight lines normal to the halfspace surface. Although this assumption agrees with observations in many cases, a complete investigation would call for allowing nonsymmetric curved crack paths and adressing also the question of the direction of propagation, which represents a much more difficult problem.

Owing to symmetry, a crack system in which the cracks remain equally long, $a_i = a_1$, is obviously one solution according to fracture mechanics. But is this solution unique, and is it stable? As shown previously [3], this is not so for the case without reinforcement if the cracks are sufficiently long relative to their spacing b. It is logical to suspect the same situation when the reinforcement is present, and to examine the effect of the amount of reinforcement on the suspected instability. We will investigate instabilities in which every other crack, of length a_2, may become longer than the intermediate cracks a_1. As shown in [3], the stability conditions for the parallel crack system with two alternating crack lengths (two interacting cracks) in a homogeneous solid reduce to the condition:

$$\frac{\partial K_2}{\partial a_2} > 0 \tag{5a}$$

and

$$\begin{vmatrix} K_1 \dfrac{\partial K_1}{\partial a_1}, & K_1 \dfrac{\partial K_1}{\partial a_2} \\ K_2 \dfrac{\partial K_2}{\partial a_1}, & K_2 \dfrac{\partial K_2}{\partial a_2} \end{vmatrix} > 0. \tag{5b}$$

It was also shown [3] that the second condition can never be violated (for admissible δa_i) if the cracks are equally long, $a_2 = a_1$. The arguments in [3] also hold in presence of reinforcement. Thus, the first condition (5a) governs in our case. As shown in [2–5], this condition may become violated in absence of reinforcement if the crack length-to-spacing ratio becomes sufficiently large [3–6]. The critical length-to-spacing ratio a_{cr}/b for which condition (5a) ceases to hold is very sensitive to the temperature or shrinkage profile [3–6]. For a given temperature profile this ratio depends only on the nondimensional parameter [5, 6]:

$$\kappa = \frac{K_c(1 - 2\nu)}{E\Delta\epsilon_{sh}^0\sqrt{b}} \tag{6}$$

where $b = 2h$ = crack spacing, and $\Delta\epsilon_{sh}^0$ = free shrinkage at the halfspace surface. (It should be noted that the use of $1 - \nu$ instead of $1 - 2\nu$ would be incorrect in case of a halfspace.) The problem of cooling is equivalent to shrinkage if the temperature profile is the same and if $\Delta\epsilon_{sh}^0 = \alpha\Delta T^0$ where a = thermal dilatation coefficient and ΔT^0 = temperature change at the surface.

If a critical state is reached, i.e. $\partial K_2/\partial a_2 = 0$ (at constant a_1), the corresponding crack length increments δa_i at constant loading (the eigenvector or the instability mode) are such that [3]:

$$\delta a_2 > 0, \quad \delta a_1 = 0, \tag{7}$$

i.e. every other crack stops growing and the remaining cracks jump ahead at no change of load. During the subsequent increase of the penetration depth D of cooling or drying, the arrested cracks gradually close, i.e. K_1 becomes less than K_c, while the leading cracks extend in a stable manner as a function of D[3]. For certain temperature profiles and an unreinforced halfspace this was shown[3] to lead to a second critical state in which the shorter cracks close, i.e. K_1 becomes zero (while $K_2 = K_c$, $a_2 > a_1$). This critical state is characterized by vanishing of the determinant (5b) and may be determined from the condition $\partial K_2/\partial a_1 = 0$ [5]. At that point the opening width of the remaining cracks a_2 at the surface is approximately twice the width that would correspond to the case of equally long cracks.

Thus, the crack instability is seen as a phenomenon determining the crack width. In reinforced concrete, we want to keep the crack width to a minimum, and so we wish the critical crack length a_{cr} at which instability occurs to be as large as possible. Therefore, we will study the effect of reinforcement on the critical crack length a_{cr}. We will pay attention only to the critical state of crack arrest ($a_2 = a_1$), which is simpler to calculate, because the critical state of crack closing (the second critical state, $K_1 = 0$, $a_2 > a_1$) follows later, and the more unfavorable bound on crack spacing and width is indicated by the first critical state.

Numerical results have been obtained by the finite element method. A typical finite element grid used is shown in Fig. 1 for a domain bounded by the lines of symmetry and extending to a depth at which the stress are negligible. The crack arrangement with unequal crack lengths after the first critical state is seen in Fig. 2. Each rectangle of the grid consists of four constant strain triangular elements and the interior node within the rectangle is eliminated by static condensation. The stress intensity factors have been calculated from the difference between the values of the strain energy contained in the grid for two adjacent crack lengths[9]. The derivatives of the stress intensity factor, $\partial K_i/\partial a_i$ have been approximated by finite difference expressions evaluated from the strain energies contained in the grid for three adjacent crack lengths[5, 6] (see Fig. 1). The method of determining the value of D for which $K_2 = K_c$ and locating the critical crack state was described in[5 and 6].

The bond between concrete and reinforcement is not perfect. As is well known, the reinforcement always slips within a certain distance from a crack which it crosses. This is clear if we realize that according to the elasticity theory the bond stress at the point where a loaded bar enters the concrete surface would be infinite. Lacking experimental information, it was assumed that the bar slips for a distance of about 1/2 in. from the concrete surface, and if we assume crack spacing of about 10 in., there is bond slip for a distance of about 0.1 h. Accordingly, the reinforcement is assumed to slip without friction in the node at the crack surface, while in all other nodes the bond is assumed to be perfect, i.e. the displacements of concrete and of reinforcement are assumed to be equal.

The profile of temperature T or free shrinkage ϵ_{sh} has been considered as the complementary error function. This function represents the exact solution of the linear diffusion equation.

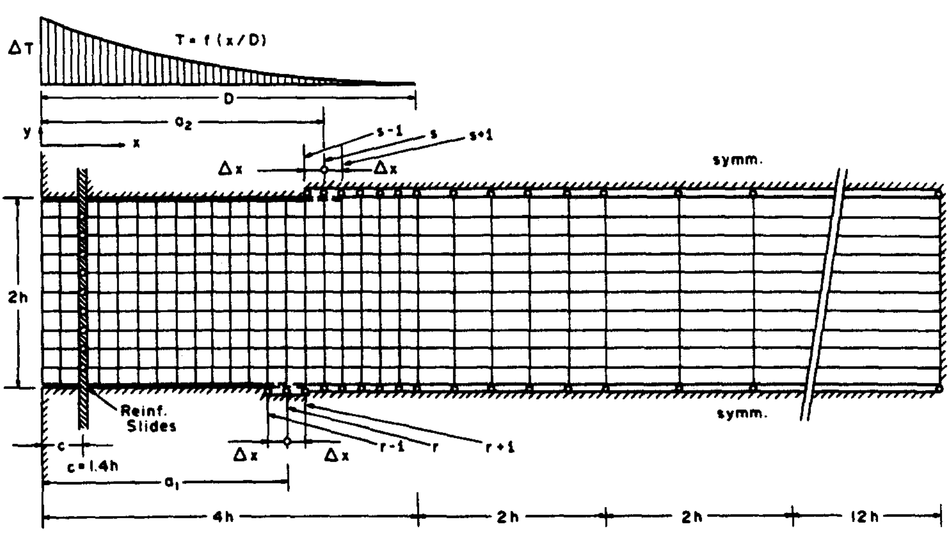

Fig. 1. Finite element grid used.

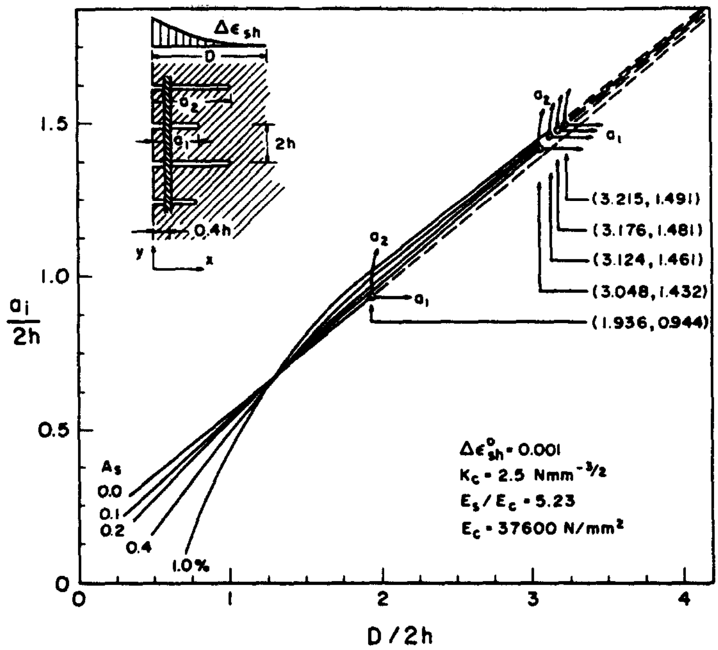

Fig. 2. Critical states and growth of shrinkage or cooling cracks for reinforcement close to surface.

The results of the calculations are plotted in Figs. 2–6. The basic material parameters and the value of free shrinkage $\Delta\epsilon_{sh}^0$ at the surface are indicated in the figures; furthermore, $\nu = 0.18$. The plots also apply to cooling cracks if $\Delta\epsilon_{sh}^0$ is replaced by $\alpha\Delta T^0$. The plots of the crack length $a_2 = a_1$ vs the penetration depth of shrinkage or cooling profile D, normalized with respect to crack spacing $b = 2h$, are shown in Figs. 2 and 3 for various cross section areas of reinforcement A_s as percentages of spacing $2h$ (both per unit length in direction z). The same results are plotted in Figs. 4 and 5 in terms of the ratio of the crack length to penetration depth. Figures 3 and 5 differ from Figs. 2 and 4 mainly in the depth of the location of reinforcement (and also in the value of $\Delta\epsilon_{sh}^0$). The stable path of the system is given by the solid lines and the unstable path, which cannot be realized for static loading, is given by the dashed lines. Also shown in the figures are the critical states (bifurcation points of the diagrams). Furthermore, the crack opening profiles of the cracks when the first critical state is reached are indicated in Fig. 6.

4. PARALLEL CRACKS DUE TO BENDING MOMENT

Consider a strip (or beam) that is subjected to pure bending and axial deformation, see Fig. 7. The axial strains before cracking, $\Delta\epsilon_b$, are then linearly distributed. Let D be the distance of

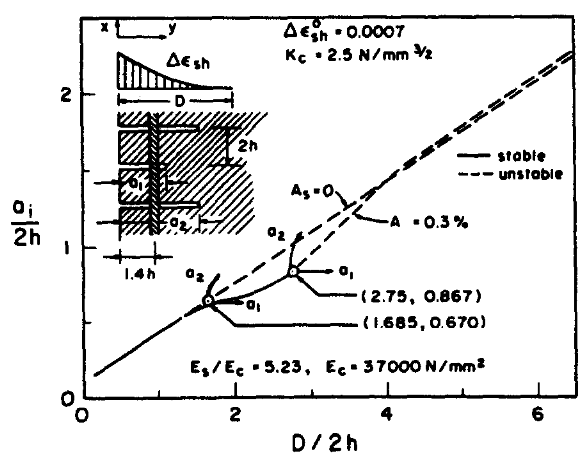

Fig. 3. Critical states and growth of shrinkage or cooling cracks for reinforcement deeper within the solid.

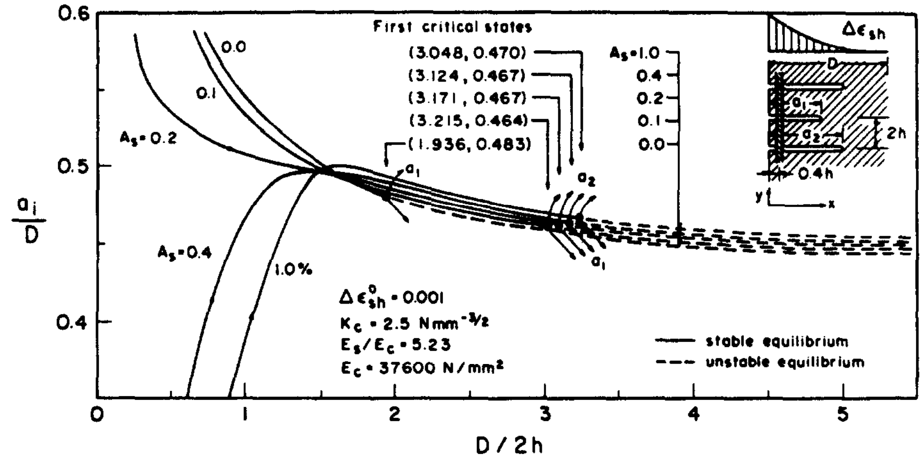

Fig. 4. Relative length of shrinkage or cooling cracks for reinforcement close to surface.

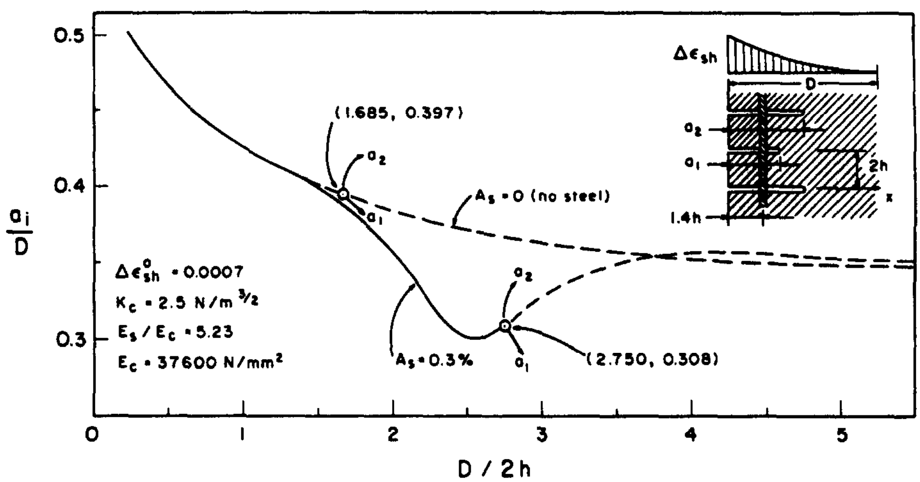

Fig. 5. Relative length of shrinkage or cooling cracks for reinforcement deeper within the solid.

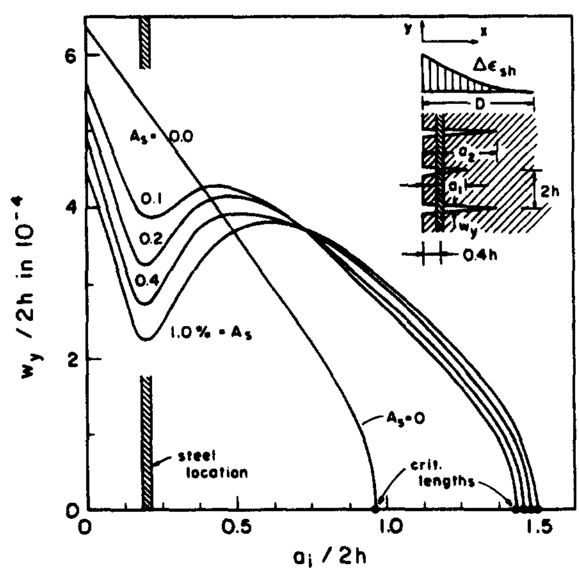

Fig. 6. Profile of the width of shrinkage or cooling crack at critical states and various reinforcement ratios.

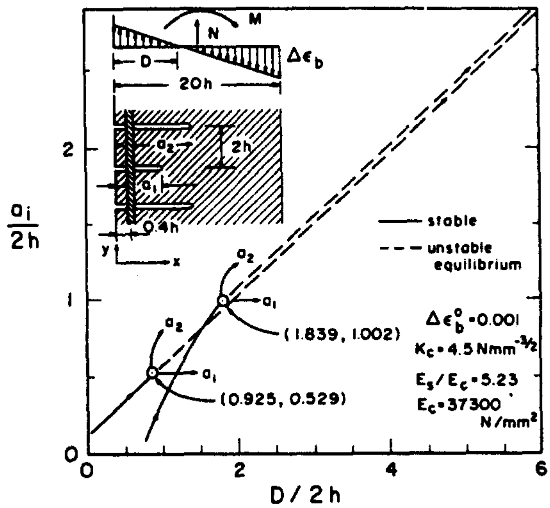

Fig. 7. Critical states and growth of bending cracks.

the neutral axis from the tensile face and $\Delta\epsilon_b{}^0$ be the value of $\Delta\epsilon_b$ at the surface. The bending moment and the axial force after cracking are denoted as M and N.

In a reinforced concrete beam subjected to pure bending, there also exist parallel equidistant cracks normal to beam surface. One purpose of the reinforcement, although not the main one, is to keep the cracks closely spaced and therefore fine. It is known that the bar size affects the crack opening in the cover of the bar, but the effect of reinforcement on the crack spacing deeper under the bar is not known. We will now see what stability consideration indicates for this problem. In agreement with observations, we will assume that the bending cracks are straight and normal to the neutral axis of the beam.

Except for the profile of strain $\Delta\epsilon_b$ and the fact that $\Delta\epsilon_b$ is a uniaxial strain whereas $\Delta\epsilon_{sh}$ is a volumetric strain, the problem is analogous to the previous one and the same solution procedure has been used. For convenience of similarity, we consider the crack length as a function of depth D to the neutral axis (Fig. 7), while strain on the surface is kept constant. Thus, the increasing D actually corresponds to a decreasing curvature and a decreasing bending moment. The finite element results are plotted in Figs. 7 and 8, in which the material properties and the value of $\Delta\epsilon_b$ are also indicated.

5. OBSERVATIONS FROM NUMERICAL RESULTS

(a) *Shrinkage and cooling cracks*

1. In a reinforced halfspace subjected to drying or cooling, an instability of the system of equidistant, parallel and equally long cracks normal to the surface may arise. If it does, every other crack stops growing and after it closes the width of the remaining cracks doubles.

2. The instability is of the same nature as found previously[2–5] for unreinforced solids. The length of equally long cracks increases as a function of shrinkage or cooling penetration depth D, and when a certain critical length a_{cr} is reached every other crack stops growing while the remaining cracks jump ahead at constant D. During the subsequent increase of D, the arrested cracks gradually close, causing the width of the remaining cracks to double.

3. The presence of reinforcement has a very significant effect on the critical crack length, a_{cr}.

4. Even a very small amount of reinforcement, such as 0.1% of crack spacing, causes the critical crack length to increase greatly, even through the effect on the crack width at the point of reinforcement is relatively small (Fig. 6). On the other hand, an increase of reinforcement from 0.1% to 1.0% has a much smaller effect and the occurrence of instability deeper inside the solid cannot be prevented by the reinforcement near the surface.

5. The reinforcement percentage of 0.1% indicated by the stability consideration is less than one half of that prescribed as a minimum shrinkage and temperature reinforcement by the American Concrete Institute (ACI) Standard 318 as well as other building codes.

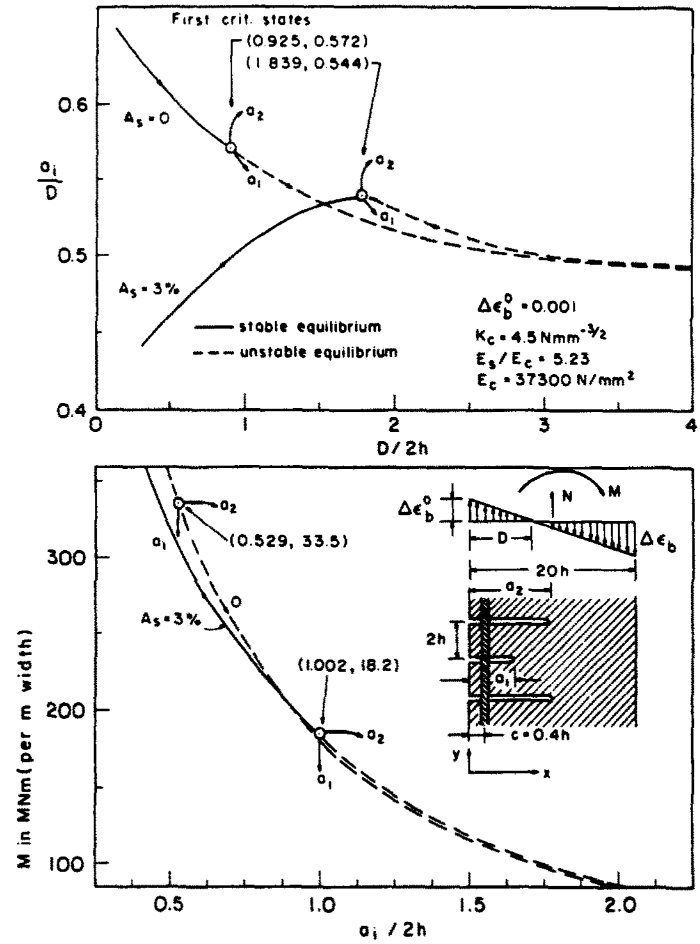

Fig. 8. Relative length of bending cracks and decline of bending moment with crack length.

6. The foregoing observations (Nos. 4 and 5) correspond, however, to the assumption of a rather short bond slip length. Otherwise, one may expect that a heavier reinforcement is needed to stabilize the cracks.

7. If the reinforcement is located at a greater depth, roughly equal to the critical crack length for the unreinforced solid, the increase of critical length due to reinforcement is about 40%, compared to 60% for a reinforcement located at a 3.5-times smaller depth. Thus, a reinforcement close to the surface seems to be more effective for suppressing crack instability near the surface and obtaining densely distributed cracks that extend deeper into the solid.

(b) *Bending cracks*

Observations numbers 1–3 also hold for bending cracks.

6. DISCUSSION OF INITIAL AND SUBSEQUENT CRACK SPACING

The present calculations do not indicate at which spacing the tensile cracks initially form. Another physical consideration is needed for this purpose. One such consideration, appropriate for the initial spacing of cracks as they start at the halfspace surface was made in[5]. This consideration may apply also to reinforced concrete, but only for cracks that are shorter than the concrete cover of reinforcement. This case is, however, not of much interest. Rather we need to know as the initial state the spacing when the cracks just reach up to the reinforcing bars or slightly beyond them. There exists a well-known physical argument which yields a lower bound on this spacing; the total tensile force that can be transmitted along segment h from the steel bars to the concrete cover by the bond stress must be equal to the tensile force resultant of stress $\sigma = f_t'$ acting in the concrete cover, f_t' being the tensile strength. This leads to a simple, well-known formula[1].

What the present stability considerations pertain to is the subsequent development of the crack spacing (and, therefore, the crack width) which gets established as the cracks grow well beyond the reinforcing bars. This determines the spacing of cracks deeper within concrete. This spacing is certainly of less interest than the spacing of cracks within the cover, but it does affect the transmission of shear forces across the rough crack surfaces and fatigue or damage due to crack movements as well as the rates at which various detrimental substances (e.g. salt) could penetrate deep into concrete.

When every other crack closes, as a result of instability, the remaining open cracks represent a crack system of the same type as that existing previously; except that the crack spacing (and width) are now doubled. Thus, the situation may repeat itself. As D further increases, a critical state of crack arrest may again be reached, after which every other among these cracks closes and the spacing of the remaining open cracks quadruples, etc.

7. CONCLUSION

Stability of the crack system is a relevant consideration for shrinkage and cooling cracks as well as bending cracks in reinforced concrete. Assuming a relatively limited bond slip, we find that the reinforcement has a profound stabilizing effect on the cracks, greatly extending the critical crack length at which the growth of every other crack gets arrested. This causes the cracks to become more densely distributed, which in turn causes a reduction of the crack width.

The reinforcement percentage that is required to achieve this is rather small, in all of our examples less than the value of 0.18%, the minimum required by ACI Code. Thus, one might feel disappointed that no new requirement to heed in design ensues from our analysis. Nevertheless, we do achieve some insight into the empirical rules for minimum reinforcement. Moreover, we find that reinforcement near the surface is incapable of enforcing a close crack spacing deep inside the solid. This is irrelevant for steel corrosion, but bears on the capability of shear stress transmission across the cracks inside the solid and the associated stiffness, dilatancy, ductility and fatigue resistance.

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†Based on Bažant's manuscript, *Stability of Crack System and Spacing of Cooling Cracks*, privately communicated to S. Nemat-Nasser and H. Ohtsubo in July 1976, and a note containing the basic formulation privately communicated to them in February 1976.