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Approximate Relaxation Function for Concrete

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Introduction

Creep properties of concrete are usually defined in terms of the creep function $J(t,t')$, which represents the strain at age t caused by a unit sustained stress acting since age $t'(t' \leq t)$. For an accurate calculation of creep effects in structures it is very useful to also know the relaxation function $R(t,t')$, which represents the stress at age t caused by a unit strain enforced at age t' and maintained constant thereafter. If we make the customary assumption that creep is linear in terms of stress and obeys the principle of superposition, the creep properties are fully defined either by the creep function or by the relaxation function, and one follows from the other. Their relationship is given by a linear Volterra's integral equation (3), which can be easily solved numerically with the help of a computer (2). For structural calculations in design offices this is, however, inconvenient and the development of some general approximate algebraic formula for inverting the creep function to the relaxation function is therefore highly desirable. Apart from that, a general algebraic inversion formula would greatly facilitate stochastic process modeling of the creep effects in structures.

For a creep function of general form, the only available algebraic inversion formula is the one based on the effective (or sustained) modulus; but the accuracy of this formula is known to be very poor (3,7) if the aging of concrete within the time interval of interest is significant, which happens in most practical cases. The same is true of special methods such as the rate-of-creep method or the rate-of-flow method (3,7,9,16), in which one simplifies the creep function to a special form that allows reduction of the problem to an easily solvable differential equation.

In 1967 Trost (17,19) proposed an approximate algebraic method that substantially reduces the error in the inversion, and a further improvement was achieved

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by a refinement of Trost's method, called the age-adjusted effective modulus method (3,4,5). In this method the classical effective modulus is corrected by a coefficient, termed the aging coefficient (4), which depends only on t and t' and can be tabulated in advance for the given creep function $J(t,t')$. This approach is, however, convenient only as long as the creep functions for different humidities and specimen sizes, plotted as functions of $t - t'$ for various t values have the same time shapes, i.e., are mutually proportional. If we would take into account that the creep curves actually have rather different time shapes depending on the humidity conditions and the size or thickness of the cross section (10) we would need a different table of the aging coefficient (or of Trost's relaxation coefficient) for each substantially different time shape of the creep curve. This would lead to impractically long tables.

The purpose of this paper is to present an approximate but quite accurate empirical formula which is generally applicable, does not require advance tabulation of any coefficient, and works well for all conceivable time shapes of the concrete creep function.

In the analysis that follows, the validity of the principle of superposition for concrete creep is assumed. This is of course not quite true (3), and there are cases of significant deviations even in the working stress range (6). It must be recognized, however, that a linear model is a necessary part of any more general nonlinear model (6).

Classical Formula

The strain ϵ at time t produced by sustained constant stress σ acting since time t' may be expressed as

$$\epsilon(t) = \sigma J(t,t') \quad (1)$$

This is often written in the form

$$\epsilon(t) = \frac{\sigma}{E_{eff}} \quad \text{with} \quad \frac{1}{E_{eff}} = J(t,t') = \frac{1 + \phi(t,t')}{E(t')} \quad (2)$$

in which E_{eff} = effective (or sustained) modulus; and $\phi(t,t') = E(t')J(t,t') - 1$ = creep coefficient (3). Although Eq. 1 is invalid when σ is variable, it is being applied in such cases as an approximation, replacing σ with $\sigma(t)$. Thus, according to Eq. 1, the stress relaxation is then approximated as

$$\sigma(t) \approx \frac{\epsilon}{J(t,t')} = E_{eff}\epsilon = \frac{E(t')\epsilon}{1 + \phi(t,t')} \quad (3)$$

which yields the classical approximate formula:

$$R(t,t') \approx \frac{1}{J(t,t')} = E_{eff} \quad (4)$$

Because $\sigma(t)$ decreases during relaxation, the actual creep is higher than $\sigma(t)J(t,t')$, and so we see that always

$$R(t,t') < \frac{1}{J(t,t')} \quad (5)$$

Eq. 4 is known to give rather accurate estimates of $R(t,t')$ if the aging is negligible (18), i.e., if $J(t,t')$ depends only on load duration $t - t'$ and not on t and t' separately. The aging is always negligible for polymers, and that is why Eq. 4 is widely used in polymer viscoelasticity. For concrete, though, the aging is significant if $t - t'$ is smaller than t', and the error of Eq. 4 is then very large, as known from computational experience (7). This is because the negative creep strain due to stress decrements after the initial loading time is reduced by aging, which means that a higher stress drop than without aging is needed to maintain a constant strain.

Necessary Corrections

In absence of aging, concrete creep depends only on load duration ξ and not on the age at load application, t'. Thus, the absence of aging is characterized by the condition $J(t' + \xi, t') = J(t, t - \xi)$ because the load duration for each of these two creep functions is the same. Therefore, the nondimensional parameter

$$\alpha_0 = \frac{J(t' + \xi, t')}{J(t, t - \xi)} - 1 \quad (6)$$

vanishes if there is no aging. Because creep for a higher age at loading is smaller, we have $J(t' + \xi, t') > J(t, t - \xi)$ if there is aging (assuming that $t' + \xi < t$). Therefore, α_0 is positive if there is aging. This suggests that Eq. 6 may be considered as a characteristic of aging.

The value of ξ must be positive and must not exceed $t - t'$ since $R(t,t')$ must not depend on the J values for times that exceed $t' + (t - t') = t$, or else the present would depend on the future. Note that for this reason, e.g., it would be incorrect to set $\alpha_0 = J(t' + \xi, t')/J(t + \xi, t) - 1$. Similarly, $R(t,t')$ must not depend on the J values for loading ages less than t' because $R(t,t')$ represents a response to stress and there was no stress before time t'. For this reason it would be incorrect to set $\alpha_0 = J(t', t' - \xi)/J(t, t - \xi) - 1$.

Since it is the aging that is responsible for the large error in Eq. 4, and since $R(t,t') < 1/J(t,t')$, the expression $1/J(t,t') - c_1\alpha_0$ in which c_1 is some constant may be expected to allow a better approximation of the relaxation function. Noting that $1/J(t,t')$ has the dimension of Newtons per square meter and α_0 is nondimensional, we divide α_0 by some value of $J(t,t')$, and the value $J(t, t - 1)$ (where time is in days) appears to give the best results.

Furthermore, noting that even in the absence of aging ($\alpha_0 = 0$) $1/J(t,t')$ is always slightly higher than $R(t,t')$, we may replace $1/J(t,t')$ with $(1 - \Delta_0)/J(t,t')$ in which Δ_0 introduces a very small age-independent correction ($\Delta_0 << 1$).

Proposed Formula for Relaxation Function

By the foregoing arguments we arrive at the following approximation of the relaxation function

$$\tilde{R}(t,t') = \frac{1 - \Delta_0}{J(t,t')} - \frac{0.115}{J(t, t-1)}\left[\frac{J(t' + \xi, t')}{J(t, t - \xi)} - 1\right] \quad (7)$$

Coefficient 0.115 has been determined by optimizing the fits of the $R(t,t')$ curves exactly calculated (from the integral equation) for various typical functions $J(t,t')$. The optimum value of ξ has been found to be

$$\xi = \frac{1}{2}(t - t') \quad (8)$$

Coefficient Δ_0 introduces a relatively minor, age-independent correction that is generally less than 0.02 and may be neglected, i.e.

$$\Delta_0 \approx 0 \quad (9)$$

More accurately, except for $t - t' < 1$ day, we may use

$$\Delta_0 \approx 0.008 \quad (10)$$

which reduces the error in $\tilde{R}(t,t')$ by about 1% of the initial value $R(28 + 0.01, 28)$.

Still more accurately, a variable coefficient Δ_0 may be used, and the following formula has been found:

$$\Delta_0 = 0.009\left[\frac{J(t' + 1, t')}{J(28 + 1, 28)}\right]^2 \frac{J(t,t') - J(t_0,t')}{J(t,t') - 0.9J(t_0,t')}; \quad \text{for} \quad t > t_0 \quad (11)$$

and $\Delta_0 = 0$ for $t \le t_0$ where $t_0 = t' + 0.01$. This formula gives $\Delta_0 \approx 0.009\,[J(t' + 1, t')/J(28 + 1, 28)]^2$ for $t - t' > 3$ days, which is independent of load duration $t - t'$ and depends only on the age at loading, t'. Eq. 11 brings about an improvement for short creep durations, such as $t - t' < 3$ days. For long-term values Eq. 11 is unnecessary.

Verification

Relaxation functions corresponding to various typical forms of $J(t,t')$ have been accurately calculated by numerically solving the integral equation with a computer, using the program listed in Ref. 2. Some of these calculations, which represent the exact $R(t,t')$, are plotted as dashed lines in Fig. 1. Figs. 1(*a*), 1(*b*), 1(*c*), 1(*d*), and 1(*f*) are based on the double power law approximations (established in Ref. 10) of the test data (13,14,15) indicated in Fig. 1. The double power law has the form (10):

$$J(t,t') = \frac{1}{E_0} + \frac{\phi_1}{E_0}(t'^{-m} + \alpha)(t - t')^n \quad (12)$$

and the values of its parameters for various data sets are listed in the figures. Figs. 1(*a*), 1(*b*), 1(*c*), and 1(*d*) pertain to basic creep, i.e., creep at constant 100% humidity, while Fig. 1(*f*) pertains to the double power law approximation (8) of drying creep data. Fig. 1(*e*) is based on the creep function

$$J(t,t') = \frac{1 + \phi(t,t')}{E(t')} \quad (13)$$

$$\text{with} \quad \phi(t,t') = 2.94t'^{-0.118}\frac{(t - t')^{0.6}}{10 + (t - t')^{0.6}} \quad (14)$$

$$E(t') = E_{28}\sqrt{\frac{t}{4 + 0.85t}} \qquad (15)$$

recommended by American Concrete Institute (ACI) Committee 209 (1,12). While the double power law, without further corrections (10), is suited for basic creep

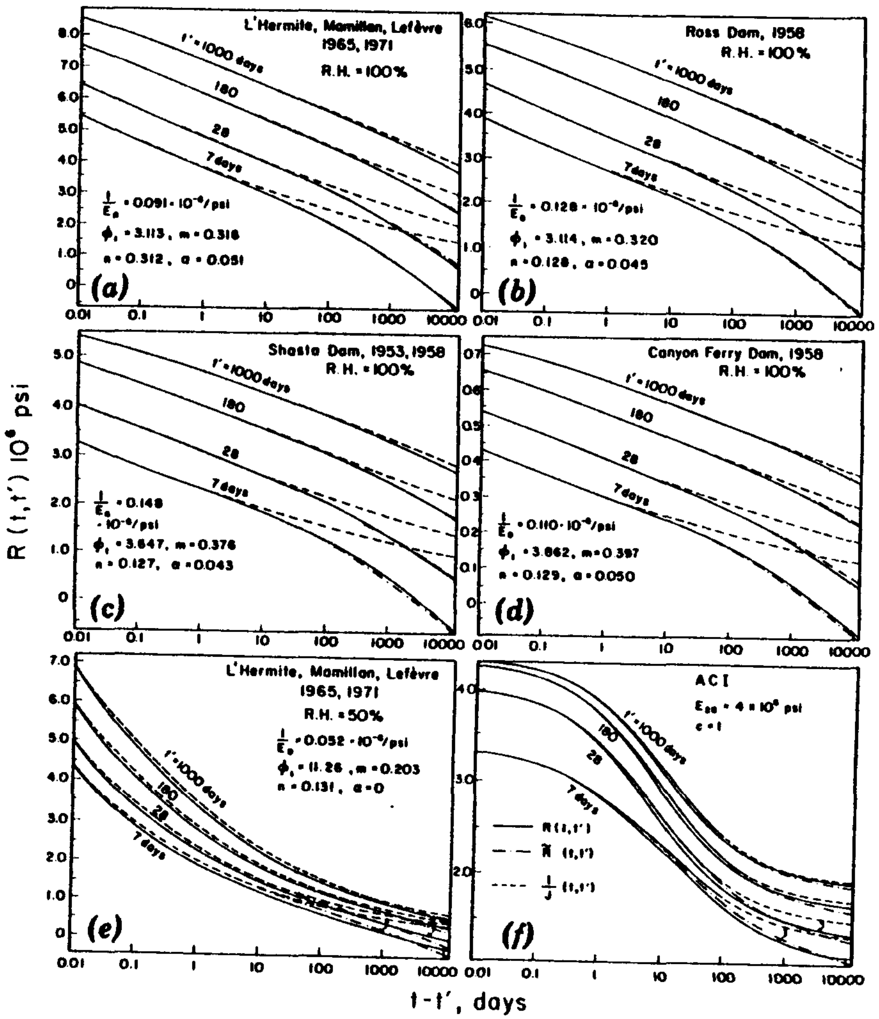

FIG. 1.—Comparison of Proposed Approximate Formula for Calculating Relaxation Function from Creep Function (-·-·-) with Exact Solution (——) and Effective Modulus Method (---) (1 psi = 6.89 kN/m²)

and for drying creep of slowly drying specimens, the ACI function is suited for drying creep of rapidly drying specimens.

The classical formula (Eq. 4, effective modulus method) is plotted in Fig. 1 as the dashed lines. We see that the proposed formula (-·-·- lines) brings about a *drastic improvement*. The errors in $R(t,t')$ relative to the initial value are also listed numerically in Table 1. Note that the proposed formula reduces the maximum error from 37% to 2% of the initial value $R(28 + 0.01, 28)$ of relaxation function at loading age 28 days. If we restrict ourselves to load durations below 3,000 days, the maximum errors are still smaller.

The formula has also been checked for creep functions of the type $1/E(t') + f(t - t') + g(t) - g(t')$, which is used in the 1978 CEB-FIP Model Code. The graphical comparisons are not shown because this formula is not a very good choice for representing concrete creep, as has been confirmed by extensive comparisons with numerous test data, which exhibited a coefficient of variation of 45% (see Ref. 10, Part VI, and Refs. 9 and 11). Nevertheless note that the proposed formula (Eq. 7) gives for this creep function far smaller errors than Eq. 4, although somewhat higher than those in Fig. 1.

Other creep functions, including empirical ones defined by a table of values, have also been checked and the proposed formula has always been found satisfactory for all practical purposes. It is observed however that the smoother

TABLE 1.—Errors Compared to Exact Solution in Proposed Formula for Relaxation Function from Creep Function, as a Percentage of Initial Exact Value $R(28 + 0.01, 28)$

			PROPOSED FORMULA					
					Maximum Error for $t - t' < 2{,}200$ days			
			$t - t' = 10^4$		$\Delta_0 \neq 0$		$\Delta_0 = 0$	
Creep data (1)	t', in days (2)	Effective modulus, $t - t' = 10^4$ (3)	$\Delta_0 \neq 0$ (4)	$\Delta_0 = 0$ (5)	$t - t'$, in days (6)	Error (7)	$t - t'$, in days (8)	Error (9)
L'Hermite et al. (1965, 1971): In water	7	31.4	0.26	0.59	—	—	—	—
	28	20.0	1.40	1.68	46	0.22	22	0.81
	180	9.0	0.48	0.73	—	—	—	—
	1,000	3.5	−0.01	0.22	464	0.19	215	0.47
Ross Dam (1953, 1958): Sealed	7	31.7	−0.44	—	22	0.12	—	—
	28	20.1	0.99	—	1,000	−0.27	—	—
	180	9.0	0.23	—	2,150	−0.18	—	—
	1,000	3.4	−0.17	—	215	0.12	—	—
Canyon Ferry Dam (1958): Sealed	7	36.9	−0.92	−0.58	10	0.11	10	0.76
	28	22.6	1.98	2.26	1,000	−0.39	1,000	0.04
	180	9.2	0.73	0.98	2,150	−0.11	—	—
	1,000	3.1	−0.03	0.20	215	0.11	215	0.39
Shasta Dam (1953, 1958): Sealed	7	36.3	−1.74	−1.39	22	0.32	10	0.71
	28	22.5	1.25	1.54	—	—	—	—
	180	9.4	0.39	0.64	2,150	−0.27	—	—
	1,000	3.2	0.03	0.05	0.5	−0.06	—	—
L'Hermite, et al. (1965, 1971): R.H. = 50%	7	11.3	−2.00	−1.92	0.1	1.78	0.1	2.39
	28	7.9	−1.56	−1.49	0.1	2.08	0.1	2.08
	180	4.5	−0.88	−0.82	0.1	2.42	0.1	2.81
	1,000	2.5	−0.11	−0.06	0.2	2.65	0.1	2.97
ACI 209 (1971): Drying	7	6.2	−0.63	—	1,000	1.50	—	—
	28	3.6	−1.32	—	22	1.86	—	—
	180	1.4	−1.47	—	22	1.77	—	—
	1,000	0.5	−0.91	—	22	1.44	—	—

the creep function, the smaller is the error. This is also apparent from the fact that the error for the drying creep cases in Fig. 1 is larger than that for the basic creep cases.

The previously available algebraic inversion based on the aging coefficient (4) would require advance tabulation of the aging coefficient for each different

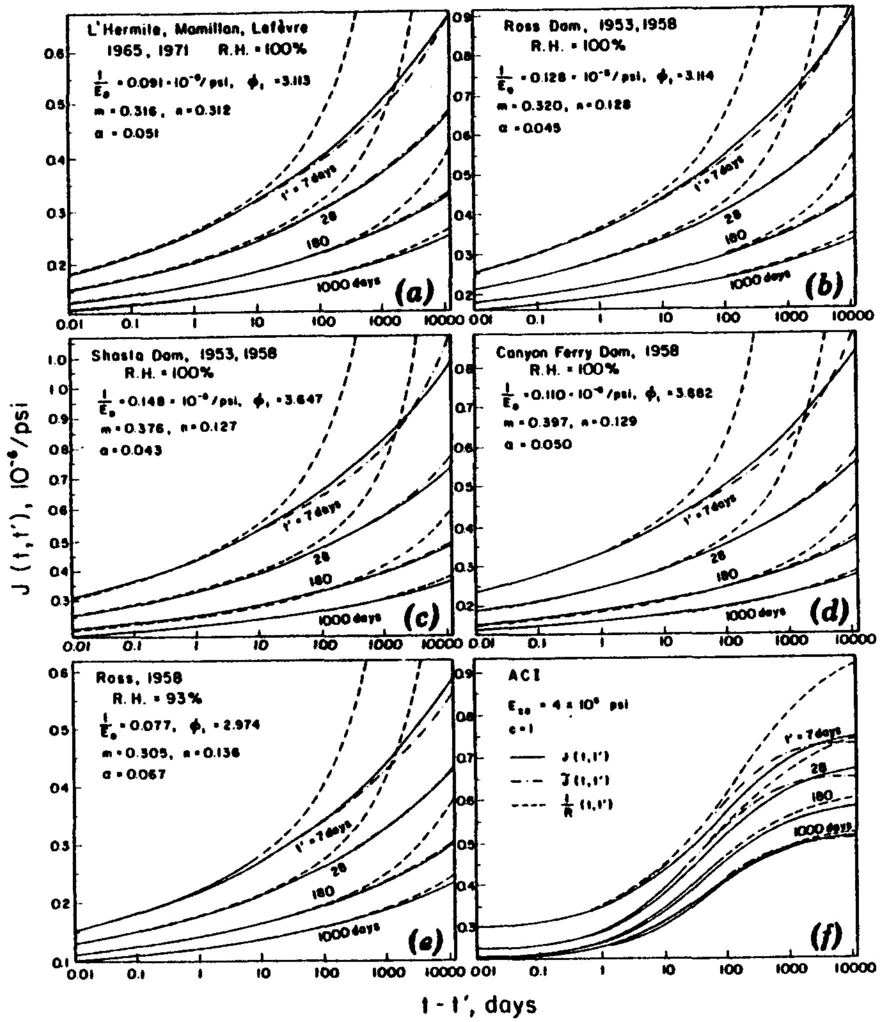

FIG. 2.—Comparison of Proposed Approximate Formula for Creep Function from Relaxation Function (Solid Lines) with Exact Solution (Dashed Lines) and Effective Modulus Method (Dash-Dot Lines) (1 psi = 6.89 kN/m²)

set of double power law parameters ϕ_1, m, n, α, or for each different set of parameters of other chosen expressions such as Eqs. 13-15. This would lead to an impractically long set of tables.

A host of papers have been devoted during the last four decades to the question of finding the best possible approximate expression for $J(t, t')$ which

would allow solving stress relaxation problems by the analytical methods for differential equations. The rate-of-creep method, rate-of-flow method (3,7), Arutiunian's creep function (7), Aleksandrovskii's creep function, current CEB Model Code formulation, etc., have all been motivated by this goal. The present

TABLE 2.—Errors Compared to Exact Solution in Proposed Explicit Formula for Creep Function from Relaxation Function, as a Percentage of Final Exact Value $J(t, t')$

Creep data (1)	t', in days (2)	Effective Modulus, $t - t' = 10^4$, in days (3)	PROPOSED FORMULA: $t - t' = 10^4$, in days (4)	PROPOSED FORMULA: Maximum for $t - t' < 2{,}200$ days: $t - t'$ (5)	PROPOSED FORMULA: Maximum for $t - t' < 2{,}200$ days: Error (6)
L'Hermite et al. (1965, 1971): In water	7	—	0.06	—	—
	28	—	2.07	—	—
	180	23.7	1.75	—	—
	1,000	5.9	1.29	1,000	−1.69
Ross Dam (1953, 1958): Sealed	7	—	3.17	—	—
	28	—	2.86	—	—
	180	22.1	1.62	—	—
	1,000	5.5	1.04	1,000	−1.55
Canyon Ferry Dam (1958): Sealed	7	—	6.59	—	—
	28	—	5.25	—	—
	180	21.7	2.35	—	—
	1,000	4.8	1.15	1,000	−2.62
Shasta Dam (1953, 1958): Sealed	7	—	8.50	—	—
	28	—	5.16	—	—
	180	21.7	2.04	—	—
	1,000	4.8	1.02	1,000	−2.29
L'Hermite et al. (1965, 1971): R.H. = 50%	7	—	3.17	—	—
	28	—	2.86	—	—
	180	22.1	1.62	—	—
	1,000	5.5	1.04	—	—
ACI 209 (1971): Drying	7	22.5	−2.56	100	4.24
	28	10.6	−2.60	100	3.36
	180	3.3	−2.04	100	2.46
	1,000	0.9	−1.02	100	1.85

general formula makes the special expressions for the relaxation functions in these methods unnecessary.

Age-Adjusted Effective Modulus

The proposed formula will be useful especially for the quasi-elastic creep analysis of concrete structures by the age-adjusted effective modulus method (3,4,5). The formula makes the tables of the aging coefficient (or Trost's relaxation coefficient, Ref. 19) unnecessary, and it in fact makes even the notion of this coefficient superfluous. Instead of calculating the age-adjusted effective modulus $E''(t, t')$ from the aging coefficient (3,4), we can now evaluate it directly in terms of the relaxation function (3,4) approximated by the proposed formula; i.e.

$$E''(t,t') \approx \frac{E(t') - \tilde{R}(t,t')}{\phi(t,t')} = \frac{1 - J(t_1,t')\tilde{R}(t,t')}{J(t,t') - J(t_1,t')} \quad (16)$$

in which $t_1 = t' + \Delta t_1$, value Δt_1 being the time interval within which the load is initially applied; $E(t') = 1/J(t_1,t')$ is the corresponding elastic modulus; and $\phi(t,t') = E(t')J(t_1,t') - 1$ is the creep coefficient. The value of Δt_1 is usually between 0.01 day and 1 day and may be taken, e.g., as 0.1 day.

It is important to note that the proposed formula for $R(t,t')$ does not involve the initial elastic modulus and does not require any short-time deformation for loads of less than 1-day duration. This feature is particularly welcome since improper matching of incompatible creep coefficient and elastic modulus has been responsible for much of the confusion and inaccuracies in creep analysis of concrete structures.

A similar feature may be observed by evaluating Eq. 16 for various chosen values of Δt_1; it is found that the long-time values of $E''(t,t')$ are rather insensitive to the chosen value of Δt_1. Thus, one is free to choose $\Delta t_1 = 1$ day, in which case, similar to $\tilde{R}(t,t')$, the values of $E''(t,t')$ are independent of the creep function for load durations less than 1 day.

It may also be noticed from calculations that the percentage error in $E''(t,t')$ is less than that in $\tilde{R}(t,t')$ and is usually below 1%, which is better than needed in practice.

Algebraic evaluation of the relaxation function and of the age-adjusted effective modulus will greatly facilitate stochastic treatment of structural creep effects. In fact, this would be hardly tractable otherwise.

Calculation of Creep Function from Relaxation Function

For this much less frequent problem, one may derive an analogous formula, using the same line of reasoning. The following approximation $\tilde{J}(t,t')$ of the creep function $J(t,t')$ has been found:

$$\tilde{J}(t,t') = \frac{1 - \Delta_1}{R(t,t') + 0.155\alpha_0(0.4 + \alpha_0)R(t,t-1)} \quad (17)$$

$$\text{with} \quad \alpha_0 = 1 - \frac{R(t'+\xi,t')}{R(t,t-\xi)}; \quad \xi = \frac{t-t'}{2} \quad (18)$$

Coefficient Δ_1 introduces a minute age-independent correction that is useful only for small load durations, such as $t - t' < 3$ days. Approximately Δ_1 may be neglected

$$\Delta_1 \approx 0 \quad (19)$$

More accurately one may consider

$$\Delta_1 = 0.01 \quad (20)$$

Still more accurately one may use variable Δ_1:

$$\Delta_1 = 0.01\left(\frac{R(28+1,28)}{R(t'+1,t')}\right)^2 \frac{R(t_0,t') - R(t,t')}{1.1R(t_0,t') - R(t,t')} \quad (21)$$

The creep function approximations according to Eqs. 17 and 18 are shown as -·-·- lines in Fig. 2. Comparing them with the exact curves (—— lines) and the classical formula $\tilde{J}(t,t') \approx 1/R(t,t')$ (--- lines), we again see a substantial improvement; the maximum error of 24% is reduced to 8% (Table 2). However, the improvement is not as spectacular as it is for the inverse problem (Fig. 1, Table 1). The main reason is that in approximating the relaxation function we seek small values from large ones, whereas here we seek large values from small ones, which is always less accurate. When $R(t,t')$ is close to 0, a small error in $R(t,t')$ is obviously translated into a large error in $1/R(t,t')$.

More accurate values of $J(t,t')$ can be calculated from $R(t,t')$ if one makes use of the previous formulas, Eqs. 7–11. First one may calculate $\tilde{J}(t,t')$ from $R(t,t')$ using Eqs. 17; then one uses this as an initial estimate, and proceeds to solve $J(t,t')$ from Eq. 7 by Newton iteration.

Summary and Conclusions

An approximate algebraic formula for calculating the relaxation function from the given creep function for aging concrete is presented. An inverse formula for calculating the creep function from the relaxation function is also found. The basic conclusions are:

1. The formula is generally applicable for all aging creep functions that may be considered for concrete creep. Thus there is no need to use special simplified forms of the creep function that allow exact calculations of the relaxation functions by differential equation solutions.
2. The error compared to the exact solution of the integral equation based on the principle of superposition is within 2% (of the initial value of stress at age 28 days). This is a drastic improvement compared to the errors up to 37% exhibited by the previously used effective modulus method.
3. Calculation of long-time relaxation as well as structural creep effects does not require knowledge of the elastic modulus and of the creep function for load durations below 1 day.
4. The formula enables direct calculation of the age-adjusted effective modulus from the creep function, and allows us dispensing with the table of the aging coefficient. This is particularly valuable when we consider a more sophisticated creep model, reflecting the fact that the creep functions for different humidities and sizes have different (nonproportional) shapes (10). Without the derived formula a different table of the aging coefficient would be required for each different shape of the creep function.

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15083 APPROXIMATE RELAXATION FUNCTION FOR CONCRETE

KEY WORDS: Aging; Approximation method; Concrete; Creep; Materials; Numerical analysis; Stress relaxation; Structural analysis; Structures; Viscoelasticity

ABSTRACT: Presented is an approximate algebraic formula for calculating the relaxation function for aging concrete. The formula is general; it applies to any form of the creep function. Compared to the previously used effective modulus method, the formula reduces the error from up to 37% to within 2% relative to the exact solution according to the superpositon principle. Calculation of long-time relaxation does not require knowledge of the elastic modulus and of the creep function for load duration below one day. The formula enables direct calculation of the age-adjusted effective modulus from the creep function, and thus it allows dispensing with the table of the aging coefficient. This is particularly useful in case of a more sophisticated creep model which reflects the fact that the creep functions for different humidities and sizes have different (nonproportional) shapes.

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