JOURNAL OF
THE ENGINEERING
MECHANICS DIVISION

Nonlinear Creep of Concrete—Adaptation and Flow

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Introduction

As proposed in 1943 by McHenry, creep of concrete under time-variable stress may be determined by the principle of superposition (5,33)

$$\epsilon(t) = \int_0^t J(t,t')\, d\sigma(t') \quad \text{. . . . . (1)}$$

in which t = time, representing the age of concrete; ϵ, σ = uniaxial strain and stress; and $J(t,t')$ = creep (compliance) function = strain at time t caused by a unit constant stress acting since time t'. Although this principle grasps the salient feature of concrete creep, significant nonlinear deviations exist. These are basically of two kinds: (a) A sustained compressive stress of low (service) level makes the response to subsequent load increments markedly stiffer, i.e., the material "adapts" itself to the load; and (b) a high stress causes a gradual weakening of the response, an intensification of creep, as if a stress-dependent "flow" rate was added to the linear creep rate.

The low-stress nonlinearity (type a) has not yet been considered in creep formulations. It is usually assumed that the creep law is linear for stresses below approx 50% of strength f'_c. Yet, this assumption is good only for relaxation-type problems but is unsatisfactory for creep recovery and for all cases of decreasing strain, although not for stress relaxation at which the strain does not decrease. Neither is it too good in situations where a large and gradual increase of stress takes place, as in creep buckling.

Experimentally, though, the low-stress stiffening is well known (31), but it has not been mathematically conceived as a nonlinearity. Instead, some investigators have recently been suggesting the use of what they call "nonvirgin"

Note.—Discussion open until November 1, 1979. To extend the closing date one month, a written request must be filed with the Editor of Technical Publications, ASCE. This paper is part of the copyrighted Journal of the Engineering Mechanics Division, Proceedings of the American Society of Civil Engineers, Vol. 105, No. EM3, June, 1979. Manuscript was submitted for review for possible publication on September 19, 1978.

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superposition, which would take "consolidation" effects into account by some linear creep law. They overlook, however, that a linear equation has only one Green's function, thus allowing only one superposition method. Any deviation from superposition is a nonlinear effect and must be modeled as such. The study that follows accomplishes that for the low-stress nonlinearity.

Upon integrating by parts and noting that $1/J(t,t) = E(t)$ = Young's modulus, one transforms Eq. 1 into the well known "impulse" formulation (5)

$$\epsilon(t) = \frac{\sigma(t)}{E(t)} + \int_0^t L(t,t')\,\sigma(t')\, dt' \quad \text{. . . . . (2)}$$

in which $L(t,t') = -\partial J(t,t')/\partial t'$ = memory function. Treatment of the high stress nonlinearity (type b) has often been attempted by extending either Eq. 2, usually by replacing $\sigma(t')$ with $F[\sigma(t')]$, or Eq. 1, usually by replacing $d\sigma(t')$ with $dF[\sigma(t')]$ (1,2,3,17,18). Alternatively, for the purpose of structural calculations, the approximate first-order differential equation of the rate-of-creep method has been made nonlinear (14,15,28,29,32,40).

However, none of these three approaches allowed a comprehensive description of test data, as exhibited in the sequel. We will adopt a different approach here, starting with the history integral for the creep rate, and we will try to obtain a better model by interpreting test data and giving consideration to the physical mechanism of creep. We will primarily have the uniaxial creep law in mind.

As for the three-dimensional nonlinear creep models, the most advanced one is that recently obtained as an extension of the endochronic theory (6). It has the advantage of reducing in the limit to a very good model of the short-time nonlinear behavior, but it does not include the low-stress nonlinearity. The model presented herein will include it. On the other hand, the high-stress nonlinearity will be treated in a simpler manner, which is completely satisfactory for uniaxial stress but is certainly crude when extended to multiaxial stress.

The nonlinearity of concrete creep is important for predicting the structural behavior. Especially, reliable evaluations of the ultimate capacity of reinforced concrete columns under combined dead and live loads (14,24,25,27,28,29,30,32,40) cannot be made without realistically modeling both the low and high stress nonlinearities of creep.

Basic Form of Proposed Constitutive Relation

Plastic deformation, as well as microcracking, usually exhibit weak memory and their rate is essentially governed by the current stress $\sigma(t)$. Therefore, the aforementioned generalizations of Eqs. 2 or 1 in which the current increase of total creep due to nonlinearity is determined by previous stress, $\sigma(t')$, do not seem to be physically reasonable. In this light, it appears to be preferable to start from the expression for the creep rate, which is obtained by differentiating Eq. 1

$$\dot{\epsilon}(t) = \frac{\dot{\sigma}(t)}{E(t)} + \int_0^t \dot{J}(t,t')\, d\sigma(t') \quad \text{. . . . . (3)}$$

in which $\dot{J}(t,t') = \partial J(t,t')/\partial t$ = creep function rate.

The nonlinear uniaxial generalization that we propose is:

$$\dot{\epsilon}(t) = \frac{\dot{\sigma}(t)}{E(t_e)} + g\,[\sigma(t)]\;\dot{\epsilon}_a(t) + \dot{\epsilon}_f(t);$$

$$\dot{\epsilon}_a(t) = \int_0^t \dot{J}(t,t'_e)\,\frac{d\sigma(t')}{1 + a(t')} \quad \text{. . . . . (4)}$$

Here, $\dot{\epsilon}_f(t)$, called the flow rate, and function $g\,[\sigma(t)]$ will be used to describe the high-stress nonlinearity, which is manifested by a weakening of the stiffness. For service stress levels we may set $g \approx 1$, $\dot{\epsilon}_f \approx 0$, and what is then left is the integral expression $\dot{\epsilon}_a(t)$, which gives an increase of the stiffness. Function $a(t')$ will be used to describe the stiffening of concrete caused by the previous sustained compressive stress, an effect brought to light by many recent tests, and t_e is the equivalent hydration period (maturity) (5), which we will exploit for describing the acceleration of aging caused by the previous sustained compressive stress. Both $a(t')$ and t_e are functions of stress; they give a gradual stiffening of the response that is important at service stress levels.

The creep function $J(t,t')$ is best defined directly by experimental creep curves for different ages t' at loading. However, if those are unavailable or incomplete, a good and simple choice for creep of sealed concrete is the double power law (5,10), expressed as $[1 + \phi_1(t'^{-m} + \alpha)(t - t')^n]/E_0$, in which m, n, ϕ_1, α, and E_0 are material parameters, whose prediction for a given type of concrete is described in Ref. 12. To model the effect of various temperature and humidity levels, the double power law may be generalized (cf. Part IV of Ref. 12) as

$$J(t,t'_e) = \frac{1}{E_0} + \frac{\phi_1}{E_0}(t'^{-m}_e + \alpha)(t - t')^n \quad \text{. . . . . (5)}$$

in which $t'_e = \int \beta_T \beta_h\, dt$; and β_T, β_h = functions of temperature T and relative humidity h (5,12).

To obtain the increase of creep due to drying, it is best to superimpose upon Eq. 5 a shrinkage-like curve (12); however, as a cruder approximation, Eq. 5 may also be used but with altered values for its constants (10).

Eq. 5 also yields the conventional elastic modulus as $E(t'_e) = 1/J(t,t'_e)$ if one substitutes $t - t' = 0.1$ day (10,12).

Stiffening Due to Low-Stress Nonlinearity—Adaptation

Consideration of the adaptation of concrete to sustained compressive stress, manifested by a gradual stiffening of instantaneous and delayed responses to further stress increments, is one essential idea of the present model. The adaptation is clearly apparent from various typical deviations from linear superposition at service stress (low stress) levels. For example, superposition overpredicts the additional deformation caused by subsequent load increases, as well as the creep recovery after unloading; see Figs. 1, 2, 3, and 4(*a*) (36,16,31,22,19).

Compression-Accelerated Aging.—In a search for an explanation, it is most simple to assume that the aging of concrete is accelerated by compressive stress. The aging phenomenon in creep, i.e., the fact that the creep due to a load

applied at a later age is smaller, is caused by the hydration of cement (or solidification). Its mechanism consists both of a growth of the volume fraction of cement gel, which prevails at early ages, and of the creation of further bonds within the gel, which prevails at later ages. It is logical to expect that a compressive stress would help the creation of such bonds, by which the microstructure adapts itself to the load. The possibility of this physical mechanism is evidenced by the well known phenomenon of healing of cracks when their

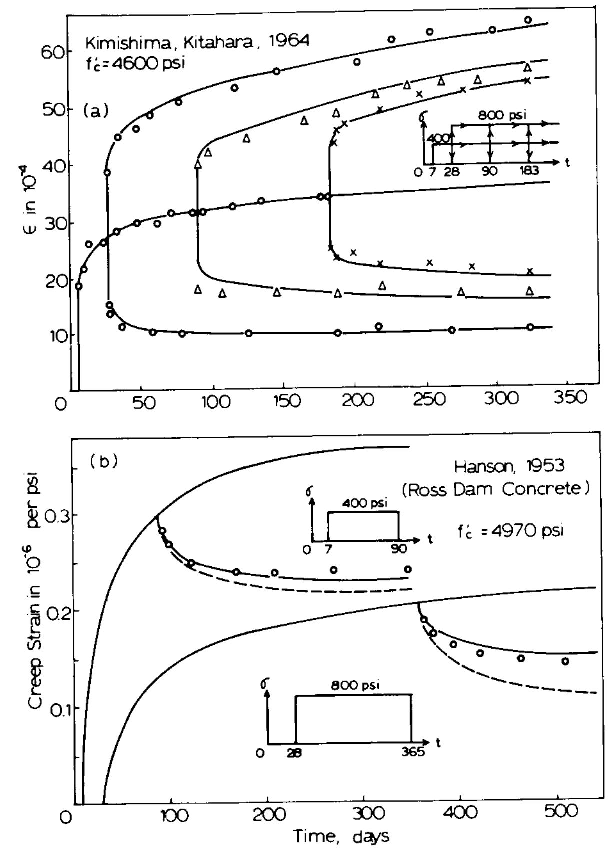

FIG. 1.—(*a*) Fit of Kimishima and Kitahara's Creep Tests with Loading and Unloading Steps; (*b*) Fit of Hanson's Creep Test Data for Ross Dam Concrete

surfaces get compressed (24) or by the gradual solidification of hydrated cement powder when subjected to hydrostatic pressure (5), as well as by the increase of compressive strength. The latter phenomenon was studied by Stöckl (39), who found the compression strength to increase with an increasing duration as well as magnitude of a compressive preload, reporting increases of 3%–18%.

The aging acceleration may be modeled by introducing the equivalent hydration period t'_e (5) as

$$dt_e = \beta_T \beta_h \beta_\sigma\, dt; \quad \beta_\sigma = 1 - a_0 I_1(t) \quad \text{. . . . . (6)}$$

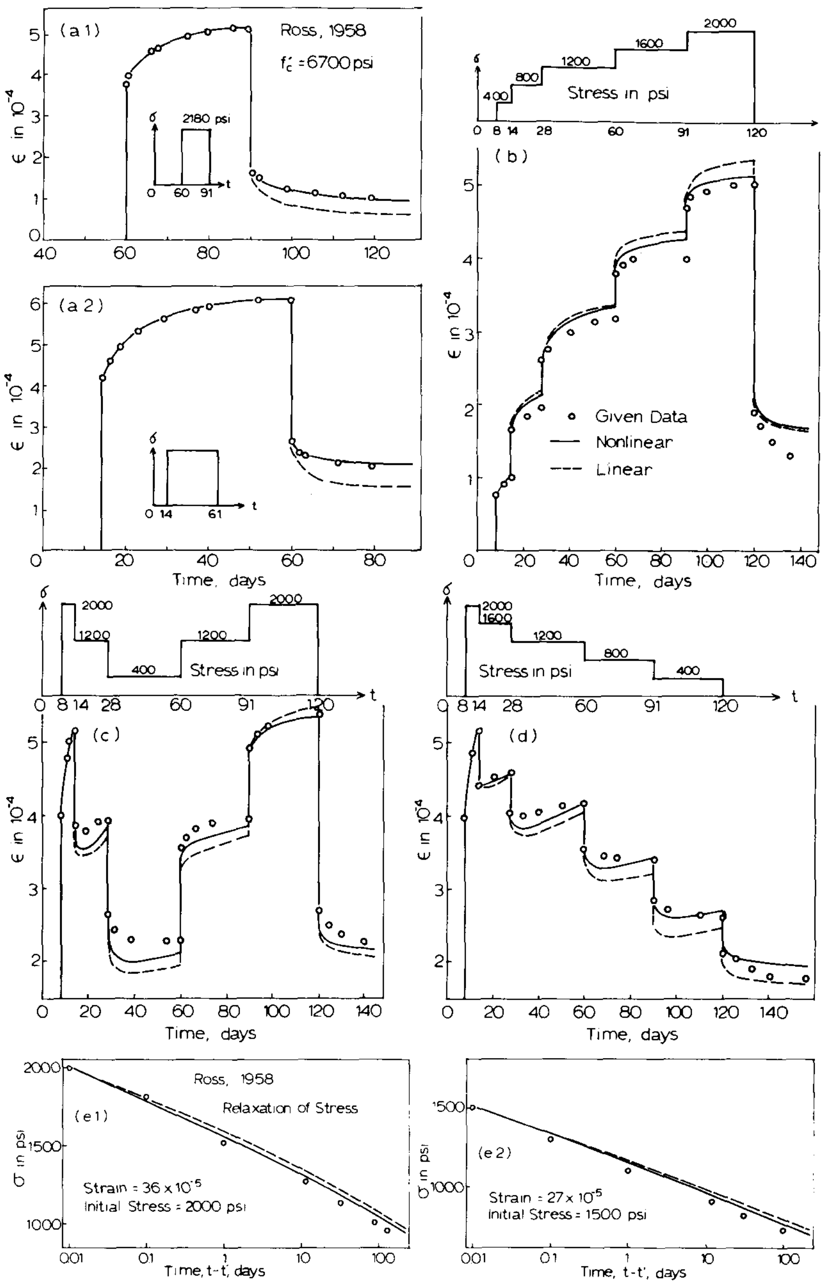

FIG. 2.—Fit of Ross's Creep Test Data; (*a*) Creep Recovery; (*b*) Stepwise Loading; (*c*) Loading and Stepwise Unloading-Reloading; (*d*) Stepwise Unloading; (*e*) Stress Relaxation

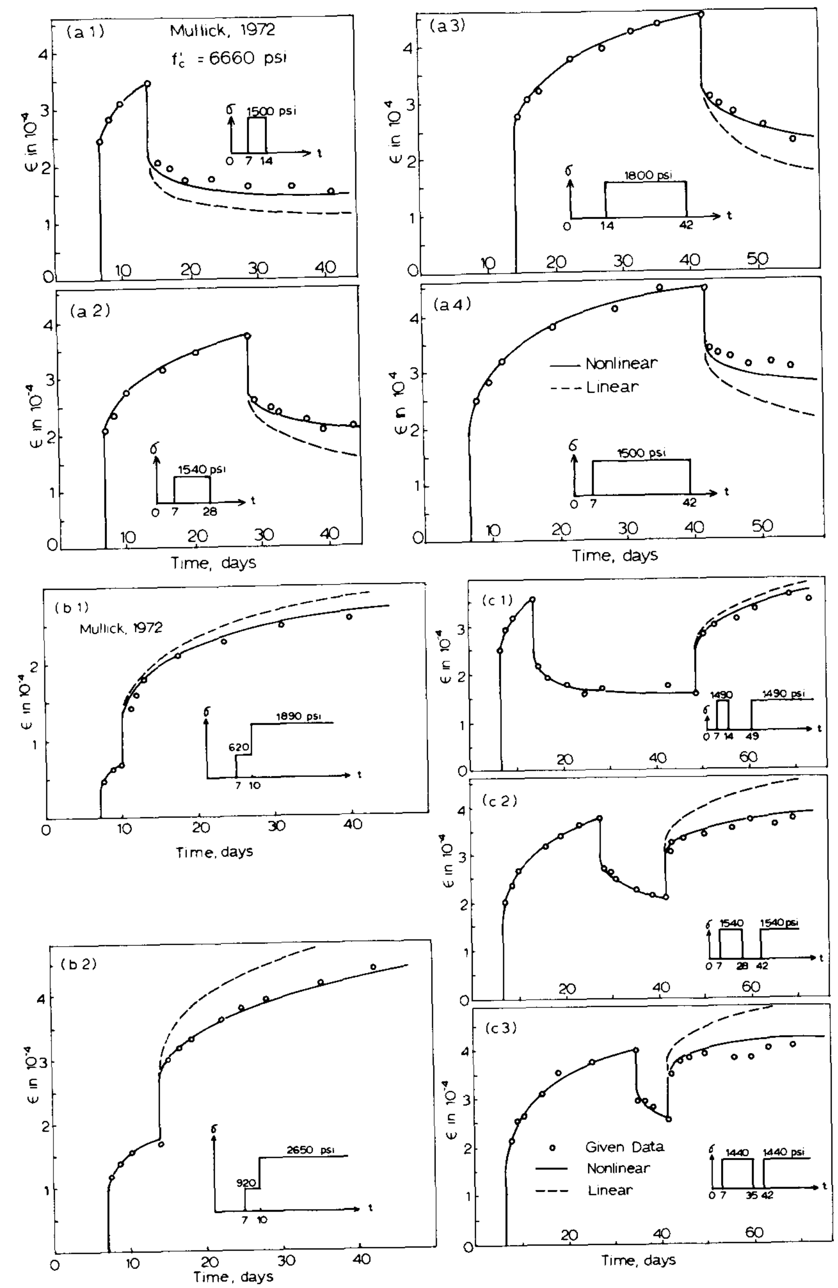

FIG. 3.—Fit of Mullick's Creep Test Data: (*a*) Creep Recovery; (*b*) Stepwise Loading; (*c*) Cyclic Loading with Different Interval Durations

in which β_T and β_h are the well known functions of temperature T and relative humidity h (5), such that $\beta_T = 1$ for 25° C, $\beta_h = 1$ for $h = 100\%$, and β_σ is a newly introduced function of stress σ. Note that Eq. 6 implies that $dt'_e = \beta_T \beta_h \beta_\sigma dt'$. Interval dt_e represents the time interval in which the number of bonds created under stress σ is the same as the number of bonds created during dt at $\sigma = 0$. For the sake of simplicity, it is assumed that the same definition of t_e and t'_e may be used for creep function $J(t, t'_e)$ and for elastic modulus $E(t_e)$.

Since function β_σ is a scalar, it can depend only on stress invariants; among these, only the first invariant, $I_1 = \sigma_x + \sigma_y + \sigma_z$ = sum of normal stresses, is suitable as a measure of compressive stress states. For uniaxial stress, $I_1 = \sigma$. The simplest dependence on I_1 is a linear one, as introduced in Eq. 6; I_1 is negative for compression, which gives $\beta_\sigma > 1$ and makes $dt_e > dt$.

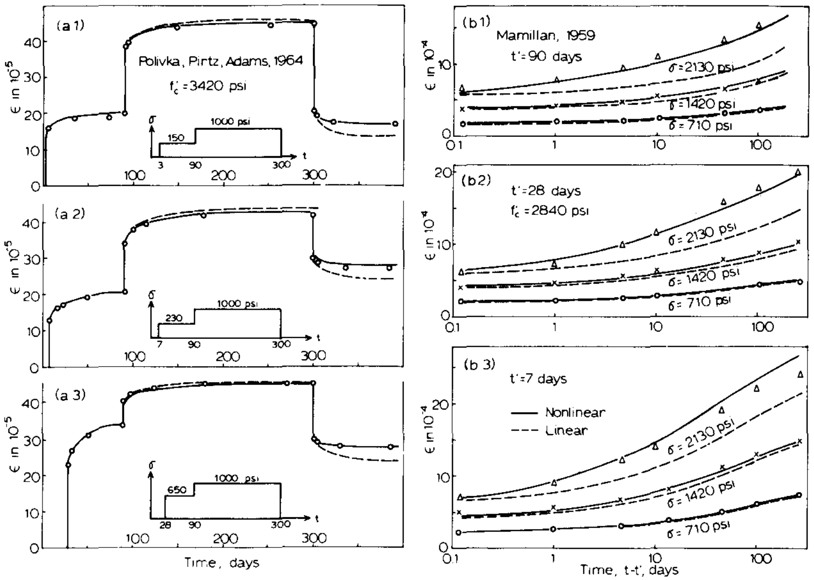

FIG. 4.—(*a*) Fit of Polivka, Pirtz, and Adams' Creep Test Data; (*b*) Fit of Mamillan's Tests of Nonlinear Creep at High Stress

Adaptation to Compression.—The amount of stiffening that can be obtained by the acceleration of aging is limited because the stiffness can never become higher than that at $t'_e \to \infty$ for concrete that is free of sustained load. Therefore, it is not surprising that, although t_e offers a distinct improvement in data fits, further compression stiffening is needed to fit test data. This may be brought about by adaptation parameter $a(t')$, which may be interpreted as the additional relative stiffness beyond that reachable by hydration acceleration, or as a measure of the number of further bonds created by compression.

Since $a(t')$ is assumed to be a scalar, it can depend only on stress invariants, of which only I_1 is again appropriate. So, we set

$$\dot{a}(t) = (a_1 + a_2 t^{-s}) \left[\frac{I_1(t)}{f'_c} \right]^r \quad \ldots \ldots (7)$$

in which f'_c = the standard 28-day cylinder strength; and a_1, a_2, s, and r = constants. The function of time, t, describes the fact that at a higher age t, when a larger number of bonds already exists, the possibility of further bond creation is reduced. Considering parameter a as a function of I_1 is motivated by the fact that no change in a occurs under pure shear stress. The initial condition for a is $a = 0$ at $t = 0$.

Undoubtedly, the healing of closed microcracks and the gradual relief of microscopic stress concentrations are also contributing mechanisms.

Eqs. 6 and 7 involve the assumption that the stiffening due to sustained stress does not cause anisotropy of the material stiffness for subsequent load increments. This is, of course, a simplification. Unless the previous sustained stress has been a hydrostatic one, these stiffnesses would in reality change differently for various directions (4). For example, for uniaxial compression creep the axial stiffness should increase but the lateral one should not, or in the case of pure shear, the stiffness should increase in one diagonal direction, but not in the other. These effects are neglected here. However, the mean picture is reasonable; e.g., in shear there should be no low stress nonlinearity in the mean, and Eqs. 6 and 7 indeed give none because $I_1 = 0$.

Weakening due to High-Stress Nonlinearity—Flow

It is more typical of inelastic behavior in general that stress causes a weakening of material stiffness. In fact, only this type of nonlinearity of creep has been considered so far. Its mechanism is progressive microcracking as well as viscoplastic slip. The former dominates at low hydrostatic pressure, and the latter at high hydrostatic pressure. Both of these phenomena exhibit in all materials a relatively weak memory, i.e., they are basically of the rate type, the rates depending on the current rather than previous stress level—a behavior that is fittingly called "flow."

Accordingly, for the purpose of a simple approximation, the flow rate, i.e., the additional creep rate that models the high stress nonlinearity in Eq. 4, may be expressed as:

$$\dot{\epsilon}_f(t) = \frac{\sigma(t) - \alpha(t)}{E_0} f[\sigma(t)]\, \dot{\phi}(t); \quad \dot{\phi}(t) = c_0 + c_1 t^{-q} \quad \ldots \ldots (8)$$

in which E_0 is introduced just for reasons of dimensionality; and $\dot{\phi}(t)$ is a function that decays with the age of concrete and models the stiffening of response with advancing hydration. Microcracking as well as plastic slip is caused chiefly by shear distorsion of concrete. Consequently, function $f[\sigma(t)]$ should increase with the second invariant of stress deviator, J_2, similar to short-time triaxial behavior (6,7,9). Since microcracking is inhibited by hydrostatic pressure, function $f[\sigma(t)]$ should also decrease with $(-I_1)$. Because at $\sigma < 0.3 f'_c$, the bond microcracking is almost nonexistent, $f[\sigma(t)]$ should be, at small stress, negligible as compared to σ; thus we may require that $f \sim \sigma^2$ at $\sigma \to 0$. These requirements are most simply modeled by

$$f[\sigma(t)] = \frac{J_2(t)}{c_2 + c_3\left(1 - \dfrac{\sqrt{3J_2(t)}}{I_1(t)}\right)} \quad \text{(9)}$$

in which $J_2 = [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]/6$, σ_1, σ_2, and σ_3 being the principal stresses. For uniaxial stress σ we have $J_2 = \sigma^2/3$, $I_1 = \sigma$, which gives $1 - \sqrt{3J_2}/I_1 = 0$; therefore, coefficient c_3 has no effect on the uniaxial behavior.

Function $\phi(t)$ gives the flow rate. This function should not be let to decrease with the load duration $t - t'$, because such a dependence could apply only to a time-constant stress and not to a variable stress $\sigma(t)$. How can we then express the fact that the creep rate decays with the load duration? This is properly done by diminishing the driving force (the stress). We may model it by function $\alpha(t)$ whose evolution may be defined as

$$\dot{\alpha}(t) = [\sigma(t) - \alpha(t)]\left[0.8 - \frac{\alpha(t)}{f'_c}\right]^3 (b_0 + b_1 t^{-p}) \quad \text{(10)}$$

with the initial value $\alpha = 0$ at $t = 0$.

With regard to three-dimensional behavior, $\alpha(t)$ may be geometrically interpreted as the current center of the loading surface in the stress space. The movement of $\alpha(t)$ is in plasticity called kinematic hardening, and its use for concrete creep is one crucial idea of the present theory. The term $\sigma(t) - \alpha(t)$ in Eq. 10 imparts to $\alpha(t)$ a tendency to always approach σ, which, in turn, causes a decay of the creep rate as seen from Eq. 8. This decay, however, does not take place at very high stress (roughly at $\sigma \geq 0.8f'_c$) (37), which is also manifested by the fact that the long-time strength of concrete is roughly $0.8f'_c$ (or $0.85f'_c$). Consequently, $\alpha(t)$ must be prevented to ever rise above $0.8f'_c$, which is achieved by the term $[0.8 - \alpha(t)/f'_c]^3$. The cube has the effect that near $\sigma = 0.8f'_c$ (e.g., for $\sigma = 0.7f'_c$) the approach of $\alpha(t)$ toward σ becomes much slower than at low σ (e.g., $\sigma = 0.3f'_c$), therefore the decay of the flow rate $\dot{\epsilon}_f$ also becomes much slower. For $\sigma \geq 0.8f'_c$, essentially a steady-state creep like that in metals is obtained. Finally, term $(b_0 + b_1 t^{-p})$ models the fact that the progress of hydration with increasing age t must cause all rates to diminish.

The flow term, $\dot{\epsilon}_f$, overshadows at high stress the integral term that alone would give a stiffening response. However, this is not enough. The integral term itself should also be allowed to soften at high stress in order to reflect the fact that the stiffening adaptation mechanism and the creation of bonds, is no doubt disrupted by microcracking due to high shear stress. Thus, it seems reasonable to multiply the integral term in Eq. 4 by a function, $g[\sigma(t)]$, which is similar to $f[\sigma(t)]$. By data fitting it has been found that

$$g[\sigma(t)] = a_3 + a_4\{f[\sigma(t)]\}^n \quad \text{(11)}$$

The function $g[\sigma(t)]$ is introduced in such a way that $g[\sigma(t)] = 1$ when $\sigma(t) = 0.3f'_c$, and 0.98 when $\sigma = 0$. So, for typical values of service stresses due to dead load, $g[\sigma(t)]$ need not be considered.

Verification, Analysis, and Ramifications

Fits of Test Data.—The constitutive relation is defined by Eqs. 4, 6, 7, 8, 9, 10, and 11. The optimum fits of the most relevant test data existing in the literature (19,20,22,23,26,31,34,35,36) obtained by numerical step-by-step integration of Eqs. 4, 6–11 (on a computer) and by optimization of the sum of square deviations from the data points in all data sets considered collectively, are

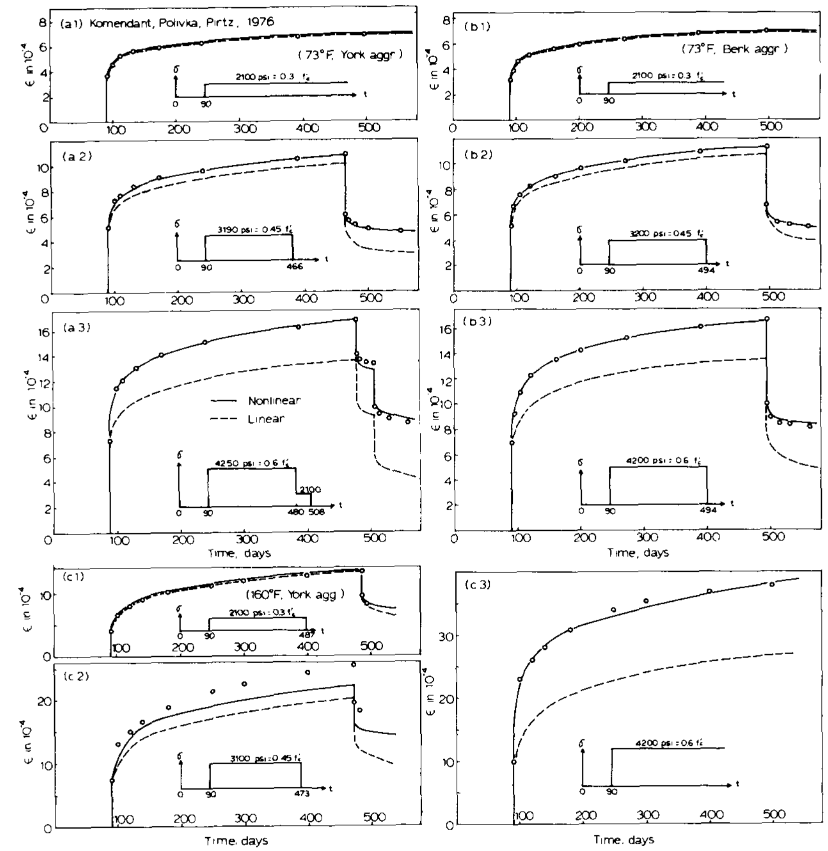

FIG. 5.—Fit of Komendant, Polivka, and Pirtz's Creep Test Data at High Stress: (*a*) Concrete with York Aggregate at Room Temperature; (*b*) Concrete with Berk Aggregate; (*c*) High Temperature Creep Test at 160° F

shown by solid curves in Figs. 1–6. They correspond to material parameters: $a_0 = 2/f'_c$; $a_1 = 0.1\ (\text{psi}/f'_c)^{1/2}$; $a_2 = 2.4\ (\text{psi}/f'_c)^{1/2}$; $a_3 = 0.98$; $a_4 = 0.104$; $b_0 = 10$, $c_0 = 0.5$; $c_2 = (0.3\ f'_c)^2$; $n = 1.5$; $r = 0.5$; $s = 0.5$; except that $a_1 = 0.2\ (\text{psi}/f'_c)^{1/2}$ for Fig. 1(*a*); $a_1 = 1\ (\text{psi}/f'_c)^{1/2}$; and $a_2 = 10\ (\text{psi}/f'_c)^{1/2}$ for Fig. 3; and $c_0 = 0.18$ for Fig. 6(*c*); because of special moisture conditions (metallic jacket filled by water). Parameters b_1, c_1, and c_3 are set equal to zero because of the lack of necessary test data, and parameters p and q are then immaterial. For comparison, the best possible fits that can be obtained with a linear creep law (Eq. 1 or 3), are shown by the dashed lines. These lines correspond to $\alpha = \sigma$; $g = 1$; $a = 0$; $t'_e = t$.

It is seen that the present nonlinear theory is quite satisfactory and much better than the linear theory. It also agrees with the test data far better than any other nonlinear creep law proposed so far. The linear creep properties, $J(t,t')$, were considered for all fits according to the measured response curves at low stress, smoothly extrapolated to various t' where needed.

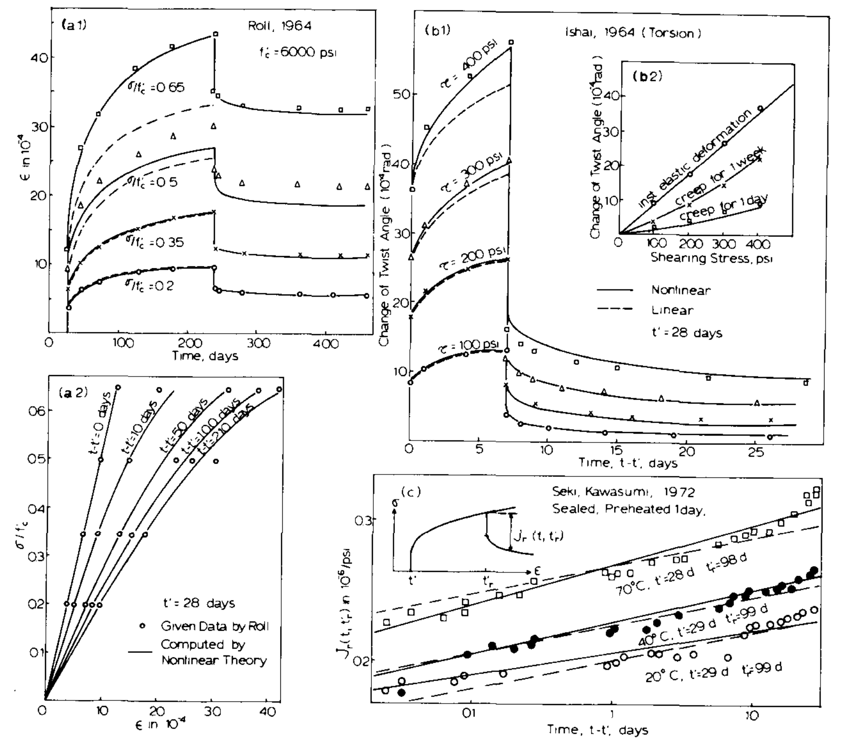

FIG. 6.—(a) Fit of Roll's Nonlinear Creep Test Data, and Its Creep Isochrones at High Stress; (b) Fit of Ishai's Torsional Creep Test Data; (c) Seki and Kawasumi's Creep Curves for Sealed Specimens Preheated for 1 day (Lines Indicate Basic Data Trend, Not Fits)

Note that our nonlinear theory gives almost the same relaxation curves [Figs. 2(*e*)] as the linear superposition. This is because the effects of $a(t')$ and t_e on one side, and of $g[\sigma(t)]$ on the other side, offset each other, and also because at the beginning of a relaxation test the stress falls very rapidly, and, therefore, $a(t')$ and t'_e do not have time to increase substantially.

The fact that the linear theory predictions are good for relaxation falls in line with the past experience. Significant differences, however, appear for creep recovery upon unloading [Figs. 1, 2, 3, 4(*a*)], which is because the sudden strain decrease does not allow $g[\sigma(t)]$ to offset $a(t')$ and t_e. For an increasing "staircase" loading (Fig. 2), the deviations from linear theory are also significant.

From Fig. 3, we see that our theory models the Mullick's effect (31), which is observed upon reloading, and consists in the fact that the recovery response gets closer to the linear superposition prediction as the preceding load period gets longer. This effect is obtained from the interplay of $a(t')$, t_e, and $g[\sigma(t)]$.

The torsional tests of Ishai (20) are fitted (Fig. 6) by transforming Eqs. 4–11 to the case of shear. For the low stress range, in which $\dot{\epsilon}_f$ is negligible, this is achieved by replacing ϵ with the shear angle γ; σ with the shear stress τ; and $J(t,t'_e)$ with the deviatoric creep function $J^D(t,t'_e) = 2(1+\nu)\,J(t,t'_e)$, in which ν = Poisson ratio ($\nu \approx 0.18$). Because $I_1 = 0$, we obtain no low stress nonlinearity in torsion ($t'_e = t'$, $a = 0$), and we see in Fig. 6 that this indeed gives good predictions of recovery. The value of f'_c in Eq. 10 has been replaced by the corresponding strength value for torsion, f'_{tor} = 600 psi (4.14 N/mm^2), for fitting these data.

On Flow Term.—It is worth noting that for constant σ, and under the assumption of constant α, the flow term $\dot{\epsilon}_f(t)$ gives $\epsilon_f \propto \phi(t) - \phi(t')$ in which t' = the time of load application and $\propto$ denotes proportionality. A term of this type has been popular with some authors, who call it "irreversible creep" or "flow" and use it as one term in the linear creep function. It has been shown, however, that a term of this type is not suitable for a linear creep law because it does not allow a satisfactory agreement with test data (11). Now we see that this may be because the term $\phi(t) - \phi(t')$ does not belong into a linear creep function. Rather, it should be considered as a nonlinear effect, as is done here.

On Creep Function Based on Recovery.—The creep function in the 1978 Comité Eurointernational du Béton (CEB) Model Code has been derived essentially from the creep curve for one loading age and from the creep recovery curve. In view of our theory, two objections may be raised against this approach. Firstly, the recovery curves should not be used as the basis for a linear creep function because the recovery is distinctly nonlinear, even though the stress relaxation follows linearity quite well. Secondly, the magnitude of the creep recovery, underlying the CEB creep function, is considered to be independent of the ages at loading and unloading, which is far from true (see Figs. 1, 2, 3, and 4 and especially Ref. (11).

The linear superposition often gives a nonmonotonic recovery curve, such that after some time the recovery slope reverses and the strain begins to increase (8). Experimentally, this is rarely observed. Calculations show that the present nonlinear model usually eliminates the recovery reversal. This points out again to the fact that the creep recovery curves cannot be predicted satisfactorily by the linear principle of superposition, and should not therefore be used as the basis of a linear creep law. The recovery does not belong to the realm of linearity, but the relaxation does.

Effect of Temperature.—Since no radical microstructural change is happening in cement paste throughout the moderate temperature range (perhaps up to 90° C), the nonlinear creep law at constant elevated temperature T should have the same form as that for the reference temperature, except that the values of some coefficients must be different. To model the temperature effect, function $\phi(t)$ (Eq. 8) was generalized as

$$\dot{\phi}(t) = (c_0 + c_1 t^{-q}) f_T \quad \text{with} \quad f_T \approx 1 + c_5(T - T_0) \quad \ldots\ldots\ldots\ldots \quad (12)$$

in which $T_0 = 25°$ C; and $c_5 = 8.3 \times 10^{-3}$. This was in addition to the temperature effect on $J(t,t')$, given by Eq. 5 (see Ref. 12). Fitting of the test data of Komendant, et al. (23) confirmed that this generalization is indeed acceptable; see the solid line fits in Fig. 5(c).

As an argument in favor of the rate-of-flow method (21), it has been recently suggested that a separation of the so-called "reversible" and "irreversible" creep is necessary to model properly the temperature effect, and it has been proposed to describe this effect by a temperature-dependent "irreversible" creep (flow) and a temperature-independent "reversible" creep (delayed elasticity). By succeeding to fit the recovery data (23) in Fig. 5 it is proven, however, that this argument lacks solid foundation. It appears unjustified to say (21) that the "reversible" creep is temperature independent. It might seem so if the data (38) shown in Fig. 6(c) [which pertains to low stress while Fig. 5(c) pertains to high stress] are approximated by parallel lines (dashed); however, we can see in this same figure that the recovery curves can be even better approximated by diverging (solid) lines (not theoretical fits). This indicates the long-time creep recovery to be rather different for different temperatures.

Three-Dimensional Generalization

Since all stress-dependent scalar parameters have already been given in terms of stress invariants, it is sufficient to replace the uniaxial σ and ϵ by the associated stress and strain deviators s_{ij} and e_{ij} and the volumetric stress and strain σ^V and ϵ^V. Thus, Eqs. 4 and 8 may be generalized as

$$\dot{e}_{ij} = \frac{\dot{s}_{ij}}{2G(t_e)} + g[\sigma(t)] \int_0^t \dot{J}^D(t,t'_e)\frac{ds_{ij}(t')}{1+a(t')} + \frac{s_{ij} - \alpha^D_{ij}}{E_0} f[\sigma(t)]\,\dot{\phi}(t);$$

$$\dot{\epsilon}^V = \frac{\dot{\sigma}^V}{3K(t_e)} + g[\sigma(t)] \int_0^t \dot{J}^V(t,t'_e)\frac{d\sigma^V(t')}{1+a(t')} + \frac{\sigma^V - \alpha^V}{E_0} f[\sigma(t)]\,\dot{\phi}(t) \quad (13)$$

in which $\sigma^V = (\sigma_x + \sigma_y + \sigma_z)/3$; $\epsilon^V = (\epsilon_x + \epsilon_y + \epsilon_z)/3$; $\dot{J}^D = 2(1+\nu)\dot{J}$ = deviatoric creep function rate; $\dot{J}^V = 3(1-2\nu)\dot{J}$ = volumetric creep function rate; $\nu \approx 0.18$ (Poisson ratio); and α^D_{ij}, α^V = centers of deviatoric and volumetric loading surfaces, whose evolution may be defined, in analogy to Eq. 10, as

$$\dot{\alpha}^D_{ij}(t) = [s_{ij}(t) - \alpha^D_{ij}(t)]\left[0.8 - \frac{\sqrt{J_2(\alpha)}}{f'_D}\right]^3 (b_0 + b_1 t^{-p})$$

$$\dot{\alpha}^V(t) = [\sigma^V(t) - \alpha^V(t)](b_0 + b_1 t^{-p}) \quad (14)$$

The uniaxial strength f'_c from Eq. 10 is replaced here by the shear strength value, i.e., the value that $\sqrt{J_2(s_{ij})}$ attains at short-time failure under the pertinent multiaxial stress. It is assumed that g, t_e, a, f, and ϕ are the same for both the deviatoric and volumetric deformations. Further, it is assumed that $J_2(\alpha)$, the second invariant of α^D_{ij}, affects only the deviatoric response, but not the volumetric one; this is not unreasonable if we realize that the arrest of hardening, modeled by the term $(\ldots)^3$ in Eq. 14, is caused by microcracking, which does not take place under hydrostatic compression. The shear strength value, f'_D, must be assumed as a function of the hydrostatic pressure $(-\sigma^V)$, as indicated by the well known Mohr's failure envelope or the Drucker-Prager failure surface (cf. the loading function in Ref. 9).

Although the triaxial generalization in Eqs. 13 is reasonable, there seem to exist no good test data on nonlinear triaxial creep that could be used to check validity quantitatively.

The previous endochronic model for nonlinear triaxial creep of concrete (6) neglected the adaptation and stiffening due to preload (described by the integral terms in Eq. 13). It accounted only for the high-stress nonlinearity (decreasing stiffness); this one was, however, modeled in a much superior way because the endochronic theory used provides a very good model of the limiting case of the short-time nonlinear triaxial deformation, while the flow terms in Eqs. 13 are too simplistic to expect from them a very good description of the triaxial deformation in short-time loading, and in consequence, the long-time loading as well. Thus, the present model and the endochronic one has each its own purpose and advantage.

Rate-Type Nonlinear Creep Law

It is well known that for the creep analysis of large structures a rate-type creep law is required (5,13). The linear integral-type law in Eqs. 1 or 3 can be approximated with any desired accuracy by the rate-type linear creep law modeled by age-dependent Maxwell chain. This law reads $\sigma = \Sigma_\mu \sigma_\mu$, $E_\mu(t)\,\dot{\epsilon} = \dot{\sigma}_\mu + \sigma_\mu/\tau_\mu$ (5,13); here $\sigma_\mu(\mu = 1, 2, \ldots n)$ = hidden stresses (internal variables); $E_\mu(t)$ = associated age-dependent elastic moduli; and τ_μ = relaxation times such that $\tau_\mu = \tau_1\, 10^{\mu-1}$ for $\mu < n$ and $\tau_n \approx \infty$.

Proceeding through the same physical arguments as before, we may now obtain the following nonlinear triaxial generalization:

$$\sigma = \sum_{\mu=1}^{n} \sigma_\mu; \quad \dot{\epsilon} - \dot{\epsilon}_f = \frac{\dot{\sigma}_\mu}{E^*_\mu} + \frac{\sigma_\mu}{\eta^*_\mu} \quad (15)$$

$$\text{with} \quad E^*_\mu = [1 + a(t)]\, E_\mu(t_e); \quad \eta^*_\mu = \frac{\tau_\mu E_\mu(t_e)}{g[\sigma(t)]} \quad (16)$$

Here $\dot{\epsilon}_f$ is given again by Eqs. 8–10, t_e is given by Eq. 6, and $a(t)$ is of the same form as given by Eq. 7 although with slightly different constants.

Summary and Conclusions

A nonlinear integral-type constitutive relation for concrete creep is developed. Interpretation of test data is aided by various microstructural arguments. Numerous test data are fitted and material parameters are identified. The conclusions are:

1. For nonlinear generalization, it is proper to start from the linear superposition integral for the creep rate rather than the total strain.
2. At low (service) stress levels, there is a significant, although previously overlooked, nonlinearity that consists in gradual stiffening or adaptation to a sustained compressive stress. It is of two kinds: (1) An acceleration of the

age-dependence of stiffness; and (2) a stiffness increase due to compressive preload. The former is modeled by a stress-dependent equivalent hydration period, and the latter by an adaptation parameter whose rate is a function of the stress and the age.

3. The high stress nonlinearity, which consists of a weakening of the stiffness, is essentially without memory and is well described by an additive rate-type flow term. Its dependence on stress and the decay of the flow rate is modeled by kinematic hardening, used so far in plasticity.

4. Our model applies at moderately elevated temperatures as well and describes adequately the creep recovery of heated concrete, even though the so-called "irreversible" temperature-independent creep component is not introduced.

5. Although the uniaxial creep is of primary interest, a rational triaxial generalization involving proper stress invariants is derived, but no test data for verification exist. For the high-stress nonlinearity, it is simpler but cruder than the previous endochronic model.

6. The neglect of the nonlinearity of adaptation to stress (which, for example, causes a reduction in creep recovery) explains the origin of certain previously found shortcomings of the "rate-of-flow" or "improved Dischinger" methods.

7. A rate-type form of the creep law is derived, but not checked by test data.

Acknowledgment

Support under National Science Foundation Grant ENG 75-14848-A01 to Northwestern University is gratefully acknowledged.

Appendix I.—Basic Information on Test Data Used

Fig. 1.—(*a*) 15-cm × 16-cm cylinders, water-cement-sand-gravel ratio: 0.42:1:1.95:3.85 by weight, sealed by neoprene tube and steel cap; (*b*) 6-in. × 6-in. cylinders, water-cement-sand-gravel ratio 0.56:1:2.75:7.14 by weight, demolded 1 day after casting and sealed.

Fig. 2.—4.625-in. × 12-in. cylinders, water-cement-sand-gravel ratio: 0.375: 1:1.6:2.8 by weight, stored at 17° C and 93% relative humidity.

Fig. 3.—4-in. × 12-in. cylinders, water-cement-sand-gravel ratio: 0.51: 1:2.2:3.3, demolded 1 day after casting and stored at 100% relative humidity and 70° F, kept saturated during test by rubber jacket filled with water around specimen.

Fig. 4.—(*a*) 16-in. × 44-in. cylinders, water-(cement + pozollan)-aggregate ratio: 0.8:1:21 by weight, cement-pozollan ratio: 0.67:0.37, cured at 70° F; (*b*) 7-cm × 7-cm × 28-cm prisms, water-cement-sand-gravel ratio: 0.49:1:1.7:3.1 by weight, cured at 99% relative humidity and 20° C for 1 day after casting, then demolded and stored at 50% relative humidity and 20° C.

Fig. 5.—6-in. × 16-in. cylinders, water-cement-sand-gravel ratios—(*a*) Berk aggregate: 0.38:1:1.73:2.6; (*b*) York aggregate: 0.38:1:1.65:2.38, sealed with butyl rubber, cured at 73° F, for high temperature tests (160° F), specimens were heated at a rate of 24° F/day up to test temperature, load applied more than 24 hr later.

Fig. 6.—(*a*) 3-in. × 10-in. cylinders, water-cement-sand-gravel ratio: 0.49:1:1.6:2.4, cured in laboratory environment for 18 hr after casting, demolded and placed for 14 days at 100% relative humidity and 70° F, then stored at 60% relative humidity and 70° F; (*b*) 5-cm × 55-cm cylinders, water-cement ratio: 0.32 by weight, cement paste-sand ratio 4:1 by volume, cured in water for 20 days, then sealed with four layers of polyethylene.

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