JOURNAL OF THE ENGINEERING MECHANICS DIVISION

# Plastic-Fracturing Theory for Concrete[a]

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## Introduction

Finite element methods make structural analysis with sophisticated constitutive relations feasible, and to obtain realistic results their use is in fact necessary. Basically, two types of constitutive relations may be distinguished: (1) Those where the relationship between stress and strain increments is linear; and (2) those where it is nonlinear (7). The former type includes the total strain theory (deformation theory) as well as hypoelasticity and incremental plasticity. The latter type includes the endochronic theory. The best available representatives of incrementally linear plastic models are those of William, et al. (1,2,41) and of Chen, et al. (16). Among hypoelastic models we can name that of Bathe and Ramaswamy (5) and among total-strain models those of Kupfer and Gerstle (28) and Cedolin, Crutzen, and dei Poli (15). All these models are simpler than the recent endochronic models (7–12). The latter ones, however, allow a much more realistic and comprehensive representation of various aspects of inelastic behavior; in particular, they give much better fits of the group of uniaxial, biaxial, and triaxial tests, provide failure conditions and strain-softening branches as part of the constitutive relation, give correct lateral strains and, most importantly, correct volume changes in all these tests and represent unloading and cyclic loading.

The present work is inspired by the previous endochronic models. In a recent comparative study of endochronic and plastic theories (8), a method of tangential linearization of the endochronic theory for proportional loading has been indicated. The resulting incrementally linearized formulation is always equivalent to some incremental plastic or fracturing model. Since most existing test data pertain to proportional or nearly proportional loading, it thus becomes clear that an

Note.—Discussion open until November 1, 1979. To extend the closing date one month, a written request must be filed with the Editor of Technical Publications, ASCE. This paper is part of the copyrighted Journal of the Engineering Mechanics Division, Proceedings of the American Society of Civil Engineers, Vol. 105, No. EM3, June, 1979. Manuscript was submitted for review for possible publication on August 17, 1978.

[a]Presented at the October 16–20, 1979, ASCE Annual Convention, Exposition, and Continuing Education Program, held at Chicago, Ill. (Preprint 3413).

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incrementally linear model that is as comprehensive and as good as the available endochronic model must be possible. This is actually accomplished in this paper, except for the time or strain-rate dependence and a high number of load cycles. The basic method of our approach has already been reported (6,7).

Our formulation has the advantage of incremental linearity, which is convenient for numerical structural analysis. However, it also has some disadvantages, i.e., the inelastic strain increments for load increments that are tangential to the loading surface are zero while the actual response should be inelastic (softer than elastic), as is obtained from the endochronic theory. Thus, in cases of material instability, predictions of the present model must be expected to give only an upper bound on the failure load (in detail, see Ref. 8).

## Incrementally Linear Inelastic Constitutive Laws

With the exception of endochronic theory, all the existing theories of time-independent inelastic behavior can be represented in the (hypoelastic) form $d\sigma_{ij} = C_{ijkm}(\underset{\sim}{\sigma}, \underset{\sim}{\epsilon})\, d\epsilon_{km}$ in which $\epsilon_{ij}$, $\sigma_{ij}$ = components of strain tensor $\underset{\sim}{\epsilon}$ and stress tensor $\underset{\sim}{\sigma}$, referred to cartesian coordinates $x_i (i = 1, 2, 3)$; and $C_{ijkm}$ = tensor of tangential moduli. Concrete is assumed to be isotropic in the initial state, but moduli $C_{ijkm}$ must exhibit stress-induced or strain-induced anisotropy. The strains may be assumed to be small.

**Plastic Deformation.**—Moduli $C_{ijkm}(\underset{\sim}{\sigma}, \underset{\sim}{\epsilon})$ represent too many unknown material functions. In incremental plasticity, the number of unknown functions is tremendously reduced through the concept of plastic loading surface (or plastic potential), defined by $F(\sigma_{km}, H_n) = 0$ in which $H_n (n = 1, 2, \ldots)$ are some state parameters, called hardening parameters. Let us begin by briefly outlining the well-known basic relations (22,30,32,34). Choosing $(\partial F/\partial H_n)\, dH_n$ to be negative for loading, and noting that $(\partial F/\partial \sigma_{km})\, d\sigma_{km} + (\partial F/\partial H_n)\, dH_n = 0$ (in which repeated indices imply summation), we have the loading criterion

$$\frac{\partial F}{\partial \sigma_{km}} d\sigma_{km} > 0 \qquad (1)$$

which represents a condition of inelastic straining. So, the plastic strain increment, $d\epsilon_{ij}^{pl}$ must be a function of this expression. Assuming linearity, we may, therefore, set $d\epsilon_{ij}^{pl} = g_{ij}(\partial F/\partial \sigma_{km})\, d\sigma_{km}$ in which $g_{ij}$ is some coefficient tensor.

Now, following Drucker (19,20), we require that the second-order work done upon applying and removing any stress increment, $d\sigma_{ij}$, be non-negative. This yields

$$\Delta W = \frac{1}{2} d\sigma_{km}\, d\epsilon''_{km} \geq 0 \qquad (2)$$

in which $\epsilon''_{km}$ = the inelastic strains = $\epsilon_{km}^{pl}$; and $\Delta W$ is shown as an area in Fig. 1(*a*). Eq. 2 is known as Drucker's stability postulate. Comparing Eq. 2 with Eq. 1, we conclude that $d\epsilon_{km}^{pl}$ must be proportional to $\partial F/\partial \sigma_{km}$, and by further comparison with the aforementioned expression for $d\epsilon_{ij}^{pl}$ we find that the proportionality coefficient must itself be proportional to $(\partial F/\partial \sigma_{km})\, d\sigma_{km}$; so

$$d\epsilon_{ij}^{pl} = \frac{\partial F}{\partial \sigma_{ij}} 2d\mu; \quad 2d\mu = \frac{1}{h}\left(\frac{\partial F}{\partial \sigma_{km}} d\sigma_{km}\right) \quad \text{(3)}$$

in which $h$ = some coefficient. The first of these relations is the famous flow rule (or normality rule) of Prandtl and Reuss (22,30,32,34).

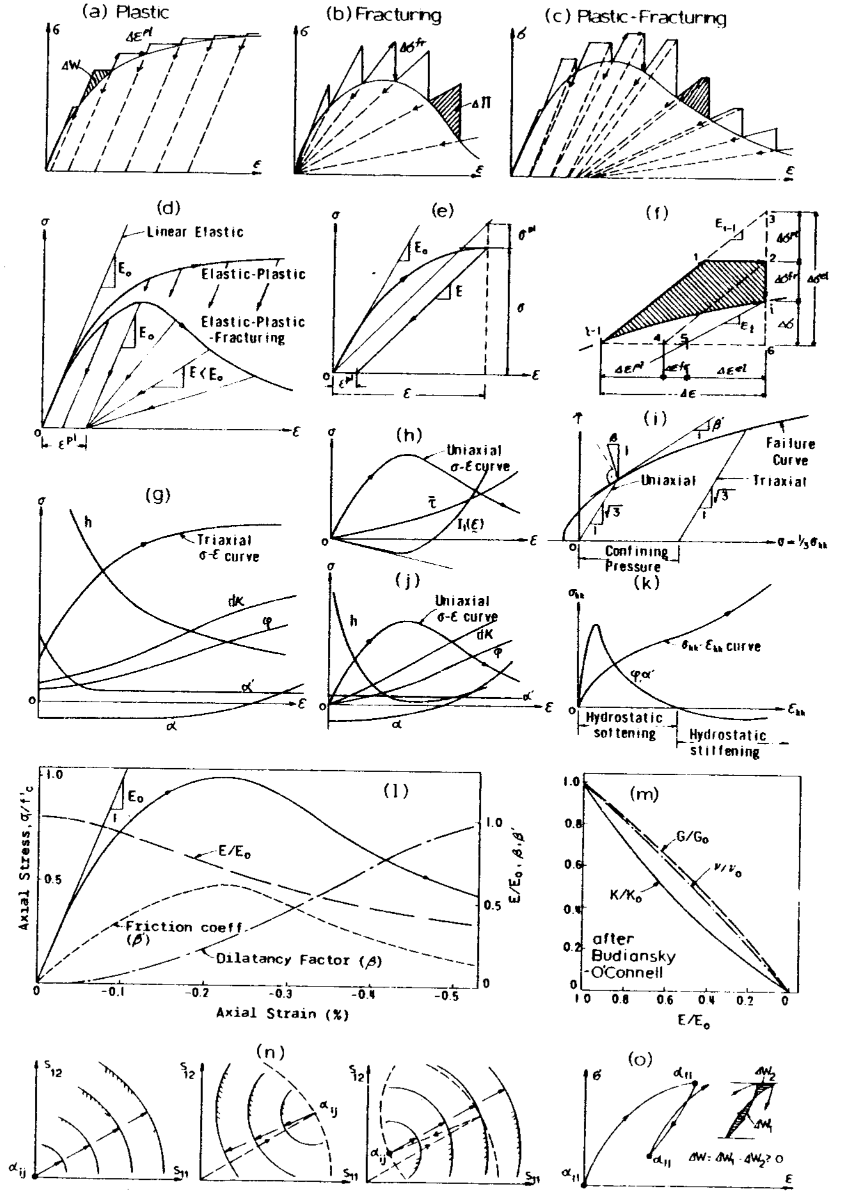


**FIG. 1.—Explanatory Diagrams and Trends of Variables**



The shape of the loading surface in the deviatoric section will be assumed to be of the von Mises type. This is no doubt a simplification, but consideration of other effects (such as fracturing strains in the sequel, etc.) appears to be much more important than playing with the shape of the loading surface. With a view toward unloading behavior, we will adopt kinematic hardening, renaming those parameters $H_k$ that give the center of the loading surface in the deviatoric and volumetric sections of the stress space as $\alpha_{ij}$ and $\alpha_v$. (For monotonic loading, though, it will be possible to use $\alpha_{ij} = \alpha_v = 0$.)

The loading function must further reflect the large effect of confining pressure $(-\sigma)$, which may be accomplished by choosing Drucker-Prager-type loading surface (21):

$$F(\sigma_{ij}, H_k) = \bar{\tau}^* + g(\sigma^*) - H_1 = 0 \quad \text{(4)}$$

in which $\bar{\tau}^* = (s_{km}^* s_{km}^*/2)^{1/2}$; $s_{km}^* = s_{km} - \alpha_{km}$; $\sigma^* = \sigma - \alpha_v$; $s_{km} = \sigma_{km} - \delta_{km}\sigma$ = stress deviator; $\delta_{km}$ = Kronecker delta; and $\sigma = \sigma_{kk}/3$ = mean stress. If $\alpha_{ij} = 0$, then $\bar{\tau}^* = \bar{\tau}$ = the stress intensity. Noting that $\partial F/\partial \sigma_{ij} = \partial F/\partial s_{ij} + \delta_{ij}\, \partial F/\partial \sigma_{kk}$ and $\sigma_{kk} = 3\sigma$, and carrying out these differentiations, one finds from Eq. 3 that

$$de_{ij}^{pl} = s_{ij}^* \frac{d\mu}{\bar{\tau}^*}; \quad d\epsilon^{pl} = \frac{2}{3}\beta\, d\mu \quad \text{(5)}$$

$$d\mu = \frac{d\bar{\tau}^* + \beta'\, d\sigma}{2h} \quad \text{if} \quad d\mu \geq 0, \quad \text{else} \quad d\mu = 0 \quad \text{(6)}$$

in which $d\bar{\tau}^* = s_{km}^*\, ds_{km}/2\bar{\tau}^*$; $\beta' = dg(\sigma^*)/d\sigma$ = internal friction coefficient; $\beta = \beta'$; $\epsilon^{pl} = \epsilon_{kk}^{pl}/3$ = volumetric (mean) plastic strain; and $e_{ij}^{pl} = \epsilon_{ij}^{pl} - \delta_{ij}\epsilon$ = plastic strain deviator. If $F$ is chosen to have the dimension of stress, coefficient $h$ may now be shown to have the meaning of plastic hardening modulus, and $\beta$ may be called the plastic dilatancy factor, for it relates the plastic volume change to the plastic deviatoric strain. The inequality in Eq. 6 ensues from the loading criterion, Eq. 1.

Eqs. 5 and 6 follow from Eq. 4 when Drucker's postulate or the normality rule is used. However, this postulate is known to lose validity when frictional deformations are present (7,19,20,33). In concrete and, similarly, in all geologic materials, the hydrostatic pressure, $-\sigma$, develops a frictional effect on inelastic shear. Consequently, the normality rule must be relaxed as far as the ratio of the volumetric deformations to the deviatoric deformations is concerned. So, the plastic dilatancy factor, $\beta$, must be considered independent of the friction coefficient, $\beta'$, i.e., the restriction $\beta' = \beta$ following from the normality rule must be relaxed. Then, of course, the normality rule is violated and $\Delta W$ in Eq. 3 can be negative, which, however, does not necessarily imply instability if the negativity of $\Delta W$ is caused by friction (6,7,33).

The total deviatoric and volumetric strains may be expressed as

$$de_{ij} = \frac{ds_{ij}}{2G} + de_{ij}''; \quad d\epsilon = \frac{d\sigma}{3K} + d\epsilon'' \quad \text{(7)}$$

in which $G$ = elastic shear modulus; and $K$ = elastic bulk modulus. Substituting Eqs. 5 and 6, $de_{ij}'' = de_{ij}^{pl}$ and $d\epsilon'' = d\epsilon^{pl}$, we now see that $ds_{ij}$ is involved in both equations linearly, i.e., $de_{ij}$ and $d\epsilon$ are expressed as linear functions of $d\sigma_{ij}$. Later, we will need an inverted form of these relations. To this end, $d\mu$ must be expressed in terms of $d\epsilon_{ij}$ rather than $d\sigma_{ij}$. To get it, Eqs. 6 and 7 may be used to calculate $d\bar{\tau}^* = 2hd\mu^* - \beta' d\sigma = 2hd\mu - K\beta'(3d\epsilon - 2\beta d\mu) = 2d\mu(h + K\beta\beta') - 3K\beta' d\epsilon$. Then, from Eqs. 7 and 5, $ds_{km} = 2G(de_{km} - s^*_{km} d\mu/\bar{\tau}^*)$, therefore, $d\bar{\tau}^* = s^*_{km} ds_{km}/2\bar{\tau}^* = (s^*_{km}/2\bar{\tau}^*)2G(de_{km} - s^*_{km} d\mu/\bar{\tau}^*)$. Equating both expressions for $d\bar{\tau}^*$, one obtains an equation that yields

$$d\mu = \frac{Gs^*_{km} d\epsilon_{km} + \bar{\tau}^* K\beta' d\epsilon_{kk}}{2\bar{\tau}^*(h + G + K\beta\beta')} \quad \ldots \ldots (8)$$

Moreover, rewriting Eq. 7 as $ds_{ij} = 2Gde_{ij} - ds^{pl}_{ij}$; $d\sigma = 3Kd\epsilon - d\sigma^{pl}$

in which $\quad ds^{pl}_{ij} = 2G\,de^{pl}_{ij}; \quad d\sigma^{pl} = 3K\,d\epsilon^{pl} \quad \ldots \ldots (9)$

are called the plastic stress decrements and, substituting Eq. 5 where $d\mu^*$ is given by Eq. 8, one can express $ds_{ij}$ and $d\sigma$ in the form of a linear function of $d\epsilon_{ij}$.

**Fracturing Deformation.**—Plastic slip does not lead to strain softening, i.e., the stress decrease at increasing strain. The only mechanism that can explain it is the microcracking or fracturing. This was realized by Dougill (18), who proceeded to make an important contribution by formulating a pure fracturing elastic material. He begins by postulating a fracturing loading surface, $\Phi(\epsilon_{ij}, H'_k) = 0$, which is assumed to envelop all states that can be reached without further fracturing. Noting that $d\Phi = (\partial\Phi/\partial\epsilon_{km})\,d\epsilon_{km} + (\partial\Phi/\partial H'_k)\,dH'_k = 0$ and choosing the second term to be negative when fracturing occurs ($H'_k$ = fracturing parameters), the loading condition is obtained as

$$\frac{\partial\Phi}{\partial\epsilon_{km}} d\epsilon_{km} \quad \ldots \ldots (10)$$

For unloading and reloading it will be useful to assume, similarly to plastic response, that the fracturing also generally exhibits kinematic hardening. The center of fracturing surface $\Phi$ in the deviatoric and volumetric sections of the strain space will be denoted as $\beta_{ij}$ and $\beta_v$ (for monotonic loading we will use $\beta_{ij} = \beta_v = 0$).

The matrix of a purely fracturing material is assumed to be perfectly elastic; therefore the material returns upon unloading to the initial stress-free state or, in the case of kinematic hardening, to the center $\beta_{ij}$, $\beta_v$. Therefore, just as in the deformation (total stress) theory, we must have $s_{ij} = 2Ge^*_{ij}$ and $\sigma = 3K\epsilon^*$ in which $G$ and $K$ are now the secant moduli, which vary during loading and are constant during unloading. Compared to hypoelastic materials, it is here not so unreasonable to assume isotropy because these relations refer to the initial stress-free state, which is isotropic. Differentiating, we obtain

$$ds_{ij} = 2Gde_{ij} - ds^{fr}_{ij}; \quad d\sigma = 3Kd\epsilon - d\sigma^{fr} \quad \ldots \ldots (11a)$$

$$ds^{fr}_{ij} = -2\,e^*_{ij}dG; \quad d\sigma^{fr} = -3\epsilon^*\,dK \quad \ldots \ldots (11b)$$

in which $e^*_{ij} = e_{ij} - \beta_{ij}$; $\epsilon^* = \epsilon - \beta_v$; and $d\sigma^{fr}$ and $ds^{fr}_{ij} = d\sigma^{fr}_{ij} - \delta_{ij}\,d\sigma^{fr}$ may be called the volumetric and deviatoric fracturing stress increments (actually decrements or relaxations). Noting that for $G = G(\epsilon_{km}, \sigma_{km})$ we have $dG = (\partial G/\partial\epsilon_{km})\,d\epsilon_{km} + (\partial G/\partial\sigma_{km})\,d\sigma_{km}$, we see that Eq. 11*a* is a special case of the hypoelastic relation $d\sigma_{ij} = C_{ijkm}d\epsilon_{km}$ in which the tangent moduli tensor, $C_{ijkm}$, is anisotropic. Thus, even though the relation between $\sigma_{ij}$ and $\epsilon$ is assumed to be isotropic, a sort of stress-induced incremental anisotropy is present. Its presence is, of course, necessary because microcracks exhibit a stress-induced preferred orientation.



From this viewpoint it is, however, clear that even the unloading behavior should be anisotropic whereas our model gives isotropic unloading behavior. This represents a simplification of the reality. For compression and shear, such a simplification seems acceptable because the orientation of cracks is not too pronounced, but the isotropy assumption would be very poor for tensile cracking, which is highly oriented. (Our study does not include the formation of continuous tensile cracks, although it does pertain to solid concrete between the tensile cracks and to concrete containing microcracks produced under a compressive stress state, which are not too strongly oriented.)

To gain a simple, manageable theory we further need some work inequality analogous to Drucker's postulate. This is provided by Il'yushin's postulate (17), which requires that the second-order complementary work upon applying and removing $d\epsilon_{ij}$ be non-negative

$$\Delta\Pi = \frac{1}{2} d\sigma^{fr}_{km}\, d\epsilon_{km} \geq 0 \quad \ldots \ldots (12)$$

Note that for the strain-softening branch we have $\Delta\Pi > 0$ while $\Delta W < 0$. This makes the fracturing theory more suitable for strain softening than plasticity. The $\Delta\Pi$ is shown as an area in Fig. 1(*b*), and the stress-strain curve is shown as a sequence of elastic and inelastic stress increments.

Because Eq. 12 indicates the occurrence of fracturing, it is necessary that $d\sigma^{fr}_{ij} = -g_{ij}(\partial\Phi/\partial\epsilon_{km})\,d\epsilon_{km}$ in which $g_{ij}$ = some coefficient tensor. Now, comparing Eq. 12 to Eq. 10, we conclude that $d\sigma^{fr}_{ij}$ must be proportional to $\partial\Phi/\partial\epsilon_{km}$, and by further comparison with the last expression for $d\sigma^{fr}_{ij}$ we find the scalar proportionality coefficient, $2d\kappa$; this yields

$$d\sigma^{fr}_{ij} = -\frac{\partial\Phi}{\partial\epsilon_{ij}} 2d\kappa; \quad 2d\kappa = \phi\frac{\partial\Phi}{\partial\epsilon_{km}} d\epsilon_{km} \quad \ldots \ldots (13)$$

analogous to Eq. 3. If $\Phi$ is chosen in a nondimensional form, $\phi$ may be called the fracturing modulus. The first of these relations is the normality rule for fracturing. Dougill (18) explored its further consequences for the degradation of the secant elastic moduli (assumed anisotropic) and for the admissible shapes of the loading surface of a path-independent material. From his systematic logical analysis it appeared that the fracturing surface must then be linear in $\epsilon_{ij}$, which corresponds to a material that consists of a random system of straight bonded fibers. This is unrealistic for concrete; therefore, the concepts of path independence and normality rule cannot be followed completely.

Analogous to Eq. 4, it is reasonable to introduce the fracturing loading surface in the form

$$\Phi(\epsilon_{ij}, H'_k) = \bar{\gamma}^* + k(\epsilon^*) - H'_1 \quad \ldots \ldots (14)$$

in which $\epsilon^* = \epsilon - \epsilon_v$, $\epsilon = \epsilon_{kk}/3$ = volumetric (mean) strain; $e_{ij} = \epsilon_{ij} -$

$\delta_{ij}\epsilon$ = strain deviator; and $\bar{\gamma}^* = (e^*_{km}e^*_{km}/2)^{1/2}$, $e^*_{km} = e_{km} - \beta_{km}$. Noting that $\partial\Phi/\partial\epsilon_{ij} = \partial\Phi/\partial e_{ij} + \delta_{ij}\,\partial\Phi/\partial\epsilon_{kk}$ and carrying out these differentiations one finds from Eq. 13 that

$$ds^{fr}_{ij} = e^*_{ij}\frac{d\kappa}{\bar{\gamma}^*}; \quad d\sigma^{fr} = \frac{2}{3}\alpha\, d\kappa \qquad (15)$$

$$d\kappa = \frac{\phi}{2}(d\bar{\gamma}^* + \alpha'\, d\epsilon) \quad \text{if} \quad d\kappa \geq 0; \quad \text{else } d\kappa = 0 \qquad (16)$$

in which $d\bar{\gamma}^* = e^*_{km}d\epsilon_{km}/2\bar{\gamma}$; and $\alpha' = dk(\epsilon^*)/d\epsilon$. However, although we formally get $\alpha = \alpha'$, we must relax this relation for the same reasons as before (Eqs. 5 and 6), assuming that, generally, the fracturing dilatancy factor, $\alpha$, is independent of the volumetric fracturing stiffness coefficient, $\alpha'$. This violates, of course, the normality rule.

To determine the degradation of elastic moduli in terms of $d\kappa$, Eq. 15 may be compared to Eq. 11*b*, and if both should hold for any $e^*_{ij}$ we must have $d\kappa/\bar{\gamma}^* = -2\,dG$ and $2\,\alpha\,d\kappa/3 = -3\epsilon^*dK$, from which:

$$dG = -\frac{d\kappa}{2\bar{\gamma}^*}; \quad dK = -\frac{2\alpha}{9}\frac{d\kappa}{\epsilon^*} \qquad (17)$$

Eliminating $d\kappa$ from Eq. 17 we may express the fracturing dilatancy factor as

$$\alpha = \frac{9\epsilon^*}{4\bar{\gamma}^*}\frac{dK}{dG} \qquad (18)$$

Eq. 18 allows us to exploit recent important theoretical results of Budianski and O'Connell (14). Using the self-consistent method for composites, they calculated the approximate macroscopic $K$ and $G$ for a perfectly elastic solid containing a random isotropic array of identical elliptical cracks of any aspect ratio $a/b$. Their results indicate a decrease of $K$ and $G$ as well as Poisson ratio $\nu$ with the crack concentration, and they are presented in the form $K/K_0 = f_K(\nu)$, $G/G_0 = f_G(\nu)$ in which $f_K$ and $f_G$ = certain monotonic functions listed in Appendix I (Eq. 29). Differentiating, we get $dK = K_0 d\nu\, df_K(\nu)/d\nu$, $dG = G_0\, d\nu\, df_G(\nu)/d\nu$, and substituting in Eq. 18 we obtain

$$\alpha = \frac{9\epsilon^*}{4\bar{\gamma}^*}\frac{K_0}{G_0}\frac{\dfrac{df_K(\nu)}{d\nu}}{\dfrac{df_G(\nu)}{d\nu}} \quad \text{with} \quad \nu = \frac{3K - 2G}{2(3K + G)} \qquad (19)$$

The dilatancy factor, $\alpha$, thus obtained does not generally obey the normality rule, of course.

The fracturing theory is more realistic than plasticity in the case of strain increments that are parallel to the current loading surface. Whereas plasticity gives in this case a purely elastic response, which is too stiff and definitely incorrect (8), the fracturing theory gives for this straining direction a stiffness that is equal to the current secant modulus and is much less than the initial



elastic modulus. (One can check it by considering an increment, $de_{22}$, from an initial state with $s_{11}$ and $e_{11}$ nonzero, all other $s_{ij}$ and $e_{ij}$ being zero.) For concrete, good data are lacking, but the equality of the tangent modulus for this load direction to the secant modulus is known to be about correct for various other materials (8). This is definitely more reasonable (and safer) than the plastic theory, which yields for the tangential strain increments the initial elastic modulus.

## Combined Plastic-Fracturing Constitutive Law

The mechanism of inelastic strain in concrete consists of both microcracking and plastic slip. The former prevails at low confining pressure and in the later stages of the uniaxial compression test. The latter dominates at high confining pressure and is also pronounced on the rising branch of the uniaxial compression test. This is shown in an idealized, exaggerated manner in Fig. 1(*d*).

The separate contributions of microcracking and plastic slip can be estimated by observing the unloading slopes. If the unloading slope is about the same as the initial loading slope, the inelastic strain is essentially plastic and is given by the offset of the unloading diagram at $\sigma = 0$ [see Fig. 1(*d*)]. If the unloading slope declines, which is typical throughout the strain-softening branch (40), the inelastic strain due to the slope decline must be attributed essentially to microcracking or fracturing, and may be defined according to Figs. 1(*e*) and 1(*f*). The fracturing behavior is obviously suitable for representing the strain softening, whereas plasticity is unsuitable for this purpose (even for obtaining a peak on the response curve), and, if used, then Drucker's postulate is violated.

The stress-strain curve may now be imagined as a sequence of elastic, plastic, and fracturing increments as shown in Fig. 1(*c*). To combine the plastic and fracturing constitutive laws, we could, in principle, proceed in two simple ways; either we superimpose $d\epsilon^{pl}_{ij}$ and $d\epsilon^{fr}_{ij}$ due to the same $d\sigma_{ij}$, or we superimpose $d\sigma^{pl}_{ij}$ and $d\sigma^{fr}_{ij}$ due to the same $d\epsilon_{ij}$. However, in case of strain softening, only the latter approach [see Fig. 1(*c*)] is admissible because we cannot associate $d\epsilon^{pl}_{ij}$ with stress decrements $d\sigma_{ij}$. So superposing $ds^{pl}_{ij}$, $d\sigma^{pl}$ (Eq. 9) and $ds^{fr}_{ij}$, $d\sigma^{fr}$ (Eq. 15), we may set

$$ds_{ij} = 2\,Gde_{ij} - 2\,G\,s^*_{ij}\frac{d\mu}{\bar{\tau}^*} - e^*_{ij}\frac{d\kappa}{\bar{\gamma}^*};$$

$$d\sigma = 3\,Kd\epsilon - 2\,K\,\beta\,d\mu - \frac{2}{3}\alpha\,d\kappa \qquad (20)$$

in which $d\kappa$ is given by Eq. 16; and $d\mu$ must be used in the inverted form (Eq. 8), which is in terms of $d\epsilon_{ij}$ rather than $d\sigma_{ij}$. The variation of $K$ and $G$ is given by Eq. 15, and $\alpha$ is evaluated from Eq. 19 on the basis of current $K$ and $G$. These equations define the constitutive relation of a plastic fracturing material, which is proposed herein.

Substituting Eq. 15 (with $d\bar{\gamma}^* = e^*_{km}de_{km}/2\bar{\gamma}^*$) and Eq. 8, as well as $de_{ij} = d\epsilon_{ij} - \delta_{ij}d\epsilon_{kk}/3$ and $d\epsilon = d\epsilon_{kk}/3$, into Eq. 20, and expressing $d\sigma_{ij} = ds_{ij} + \delta_{ij}d\sigma$, we can bring the constitutive relation to the form

$$d\sigma_{ij} = C_{ijkm}\, d\epsilon_{km} = \left[ C^{el}_{ijkm} - \frac{\left(\dfrac{G}{\bar{\tau}^*} s^*_{ij} + K\beta\delta_{ij}\right)\left(\dfrac{G}{\bar{\tau}^*} s^*_{km} + K\beta'\delta_{km}\right)}{G + h + K\beta\beta'} \right.$$

$$\left. - \phi\left(\frac{e^*_{ij}}{2\bar{\gamma}^*} + \frac{\alpha}{3}\delta_{ij}\right)\left(\frac{e^*_{km}}{2\bar{\gamma}^*} + \frac{\alpha'}{3}\delta_{km}\right)\right] d\epsilon_{km} \quad \text{. . . . . . . . . . . . . . . . (21)}$$

with $$C^{el}_{ijkm} = G(\delta_{ik}\delta_{jl} + \delta_{jk}\delta_{il}) + \left(K - \frac{2}{3}G\right)\delta_{ij}\,\delta_{kl} \quad \text{. . . . . . . . . . . . (22)}$$

in which $C^{el}_{ijkm}$ = the current isotropic tensor of elastic moduli; and $C_{ijkm}$ = the tensor of tangent moduli (needed for a finite element program). Note that tensor $C_{ijkm}$ is of a form that exhibits both stress-induced and strain-induced anisotropy and is nonsymmetric unless $\alpha = \alpha'$ and $\beta = \beta'$. Eq. 21 applies only for $d\mu \geq 0$ and $d\kappa \geq 0$. If $d\mu < 0$, then according to plasticity theory one should set $h \to \infty$, and if $d\kappa < 0$, one should set $\phi = 0$; however, more complicated rules are needed to reflect the real behavior, as will be seen in the sequel.

In the definition of the fracturing strain increments in the presence of plastic strain there is, from the theoretical point of view, a certain ambiguity. It may seem equally logical to exclude the plastic strain from the definition of $\Phi$. In that case, Eqs. 11*b* and 15 would change to $ds^{fr}_{ij} = -2(e_{ij} - e^{pl}_{ij})dG = (e_{ij} - e^{pl}_{ij})d\kappa/\bar{\gamma}^*$, $d\sigma^{fr} = -3(\epsilon - \epsilon^{pl})dK$. Similarly, $\alpha$ could be defined as $\alpha = [g(\epsilon - \epsilon^{pl})/4(\bar{\gamma} - \bar{\gamma}^{pl})]\, dK/dG$. Eqs. 11*b*, 15, and 18 are used here in the presence of plastic strain because they appear to give better fits of test data.

The constitutive law defined by Eq. 21 or by Eqs. 20, 8, 16, 15, and 19 involves, aside from the initial elastic constants, $G_0$ and $K_0$, six independent scalar coefficients, $h$, $\phi$, $\beta'$, $\alpha'$, $\beta$, and $\alpha$, characterizing the material. These coefficients are functions of the invariants of stress and strain (see Appendix I). One of these functions, i.e., $\alpha$, has been determined by a theoretical argument based on a micromechanics model (Eq. 19). However, for the other five functions, no micromechanics models are available at present, and, therefore, $h$, $\phi$, $\beta'$, $\alpha'$, and $\beta$ had to be determined empirically, by fitting test data. The selection of the form of these functions has been guided by various physical as well as intuitive concepts (see Appendix I).

For data fitting, a computer program similar to that used previously for the endochronic theory (listed in Ref. 10) has been developed. The program uses small loading steps to integrate the constitutive relation, Eq. 19, for a specified form of functions $h$, $\phi$, $\beta'$, $\alpha'$, and $\beta$. The numerical algorithm is analogous to that in Ref. 9, and the simulation of various types of tests (uniaxial, biaxial, triaxial, shear compression, etc.) is done in the same manner. The response curves are automatically plotted by a Calcomp plotter, and a sum of the square deviations from the characteristic data points (the objective function to be minimized) is evaluated. The data fitting is done collectively for various data sets for various concretes, searching simultaneously for the dependence of the material parameters on the strength of concrete. (For details see Ref. 10.)



The representative test data for monotonic loading that are available in the literature (3,4,24–29,35,37) have been fitted as shown in Figs. 3–5 (with $\alpha_{ij} = \beta_{ij} = \alpha_v = \beta_v = 0$) where the fits by the present theory are indicated by solid lines. The expressions for material functions that correspond to these fits are listed in Appendix I (Eqs. 30–34). It must be emphasized that all fits correspond to the same set of 26 material parameters given as functions of the 28-day standard cylindrical strength $f'_c$ (Eq. 33). Needless to say composition parameters of concrete would have to be taken into account to achieve better fits.

### Jump-Kinematic Hardening for Unloading and Cyclic Loading

Strictly withn the spirit of the theories of plasticity and fracturing, unloading as well as reloading would be perfectly elastic. This is of course not at all true for concrete. One reason for the astonishing success of the endochronic theory was just the fact that unloading and reloading are inelastic in that theory. As already mentioned, by tangential linearization the endochronic theory can be converted to an incrementally linear form (8), which is, in the case of proportional loading, generally equivalent to a plastic-fracturing formulation except that it does give inelastic strain for unloading and reloading. Thus, noting that the intrinsic time of endochronic theory is analogous to our parameters $\mu$ and $\kappa$, we may try to modify the definition of $\mu$ and $\kappa$ so as to have positive $d\mu$ and $d\kappa$ for unloading and reloading and thus obtain inelastic strains.

With regard to cyclic loading, the question of Drucker's postulate (Eq. 2) deserves closer scrutiny. In the presence of friction, this postulate of course need not and should not be satisfied. However, restricting attention to the pure shear strains (deviatoric strains) at $\sigma = 0$ we have no friction, and thus no reason for violating this postulate. The early version of endochronic theory (9,11) did violate it nonetheless for the pure shear as well, as far as infinitesimal unload-reload cycles are concerned (8). Recently, though, a modification that can satisfy this postulate in all cases ($\Delta W \geq 0$) yet allows for inelastic strain during unloading and reloading has been proposed (section 13 of Ref. 8).

The essential idea, which will now be adapted to plastic-fracturing materials, is to introduce kinematic hardening such that the center of the current loading (or fracturing) surface ($\alpha_{ij}, \alpha_v, \beta_{ij}, \beta_v$) is allowed to jump instantaneously into the current stress point (or strain point) in the stress (or strain) space as soon as loading reverses into unloading (8). When unloading reverses into reloading, again the center is let to jump into or below the current stress (strain) point, etc. This is sketched in Fig. 1 (*n*,*o*), in which we see that the state point never moves back inside the loading surface; it can only move outward, whether we have loading or unloading [Fig. 1(*n*)].

Thus, the loading surface now looses the meaning of an envelope of all states that are elastically attainable. In fact, if an associated loading-unloading criterion was used, no state would be elastically attainable.

The question of the unloading-reloading criterion must now be reexamined. Obviously, there is no compelling reason to associate the loading criterion with the loading surfaces since the normality rule is generally violated because of friction ($\beta' \neq \beta, \alpha' = \alpha$). Surface $F(\sigma_{ij}, H_k)$ is not suitable as the unloading criterion because it shrinks during strain softening (declining stress-strain diagram), which may not be confused with unloading. Fracturing surface $\Phi(\epsilon_{ij}, H_k)$ would be acceptable in this respect, but its use would be illogical for the plastic parts of strains. Thus, other criteria have been searched, and it appeared that the signs of the internal volumetric work, $W$, and deviatoric work, $W'$, per unit volume, defined as

$$dW = 3\sigma d\epsilon; \quad dW' = s_{km} de_{km} \qquad (23)$$

provided a simple and realistic distinction between loading and unloading (8).

We may now introduce loading-unloading-reloading criteria as follows:

For $dW \geq 0$ and $W = W_0$: $c_1 = 1$; $dG$ and $\alpha$ from Eqs. 17 and 19 (24*a*)

For $dW' \geq 0$ and $W' = W_0'$: $c_1' = 1$; $dK$ from Eq. 17 . . . . . . . . (24*b*)

For $dW < 0$: $c_1 = c_u$; $G$ and $\alpha$ from Eq. 26 . . . . . . . . . . . . . (24*c*)

For $dW' < 0$: $c_1' = c_u$; $K$ from Eq. 26 . . . . . . . . . . . . . . . . (24*d*)

For $dW \geq 0$ and $W < W_0$: $c_1 = c_r$; $dG$ and $\alpha$ from Eqs. 17 and 19 (24*e*)

For $dW' \geq 0$ and $W' < W_0$: $c_1' = c_r$; $dK$ from Eq. 17 . . . . . . . . (24*f*)

Here, $W$ and $W_0'$ are the maximum values of the deviatoric and volumetric work attained up to the current load step, and coefficient $c_1$ is used to redefine Eqs. 8 and 16:

$$d\mu = \frac{c_1 G s^*_{km} d\epsilon_{km} + c_1' 3\bar{\tau}^* K\beta' d\epsilon}{2\bar{\tau}^*(h + G + K\beta\beta')};$$

$$d\kappa = \frac{\phi}{2}\left(c_1 \frac{e^*_{km} d\epsilon_{km}}{2\bar{\gamma}^*} + c_1'\alpha' d\epsilon\right) \qquad (25)$$

However, if we obtain $d\mu < 0$, we must set $d\mu = 0$, and if $d\kappa < 0$ we must set $d\kappa = 0$. This restriction is necessary because our criteria in Eqs. 24 are not associated with $d\mu$ and $d\kappa$ and for some straining that is nearly parallel to the loading surface, negative $d\mu$ or $d\kappa$ could conceivably arise, which is inadmissible.

For unloading, the elastic properties are empirically defined as

$$K = K_p\left(1 - 0.6\frac{\sigma}{f_c'}\right); \quad G = G_p\left(1 - 1.8\frac{\sigma + I_3^{1/3}}{f_c' - 0.1\sigma}\right); \quad \alpha = \frac{9\epsilon^*}{4\bar{\gamma}^*}\frac{dK_p}{dG_p} \qquad (26)$$

in which $K_p$, $G_p$ = values of $K$ and $G$ at the previous peak point of $W_0$ or $W_0'$, respectively; and $I_3$ = absolute value of the third invariant of $\sigma_{ij}$. Derivative $dK_p/dG_p$ in Eq. 17 for $\alpha$ is taken as the value of $dK/dG$ at the previous peak point of $W_0$.

The kinematic hardening is defined as follows:

When $dW$ changes from $\geq 0$ to $< 0$, set $\alpha_{ij} = s_{ij}$; $\beta_{ij} = e_{ij}$ . . . . . . (27*a*)

When $dW'$ changes from $\geq 0$ to $< 0$, set $\alpha_v = \sigma$; $\beta_v = \epsilon$ . . . . . . . (27*b*)

When $dW$ changes from $< 0$ to $\geq 0$, set $\alpha_{ij} = s_{ij}$; $\beta_{ij} = \frac{e_{ij}}{2}$ . . . . . . (27*c*)



When $dW'$ changes from $< 0$ to $\geq 0$, set $\alpha_v = \sigma$; $\beta_{ij} = \frac{\epsilon}{2}$ . . . . . . (27*d*)

Various details in the foregoing method for unloading and reloading are strictly empirical, obtained from the fitting of tests. Coefficient 2 in Eqs. 27*c* and 27*d* means that for reloading, the center of the fracturing surface is not placed into the last lower limit point but between this point and the origin. Eq. 26 means that during unloading, the elastic moduli are not calculated according to the pure fracturing material (which would require $dG = dK = 0$). At high

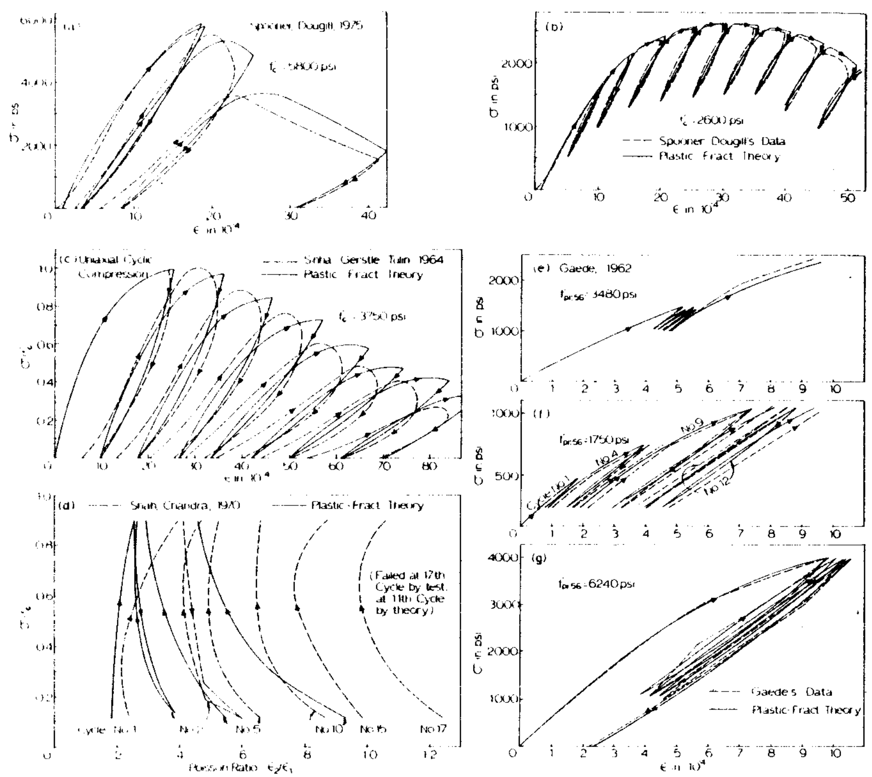

FIG. 2.—Fits of: (*a*–*g*) Uniaxial Cyclic Tests with (*d*) Lateral Strain Ratio

confining pressures ($\sigma << 0$), Eq. 26 gives a much larger value of $K$ than that for the loading branch, which is needed mainly for fitting the steep unloading branch under hydrostatic loading [Fig. 3(*b*)]. Physically, this is not unreasonable because high confining pressures should keep all voids and microcracks closed, resulting in high $K$ and $G$. During unloading, $|\sigma|$ decreases, which makes $K$ and $G$ again smaller and provides the characteristic decline of slope of the unloading branch during unloading [Fig. 3(*b*)]. The fracturing dilatancy factor, $\alpha$, is determined, however, on the basis of elastic moduli $G_p$ and $K_p$ for the last upper peak point all throughout the unloading branch.

In the basic relations, Eqs. 20, 8, 16, and 17, material functions $h$, $\phi$, $\alpha$, $\beta$, and $\beta'$ can, in general, be introduced in forms that differ for loading, unloading, and reloading. Only a simple change in these functions is considered herin as indicated by coefficient $c_1$. This coefficient models the fact that for unloading, $c_1 = c_u$, the inelastic strain is less than for reloading, $c_1 = c_r$, for which it is again less than for virgin loading, $c_u < c_r < 1$. It was proved (8) that

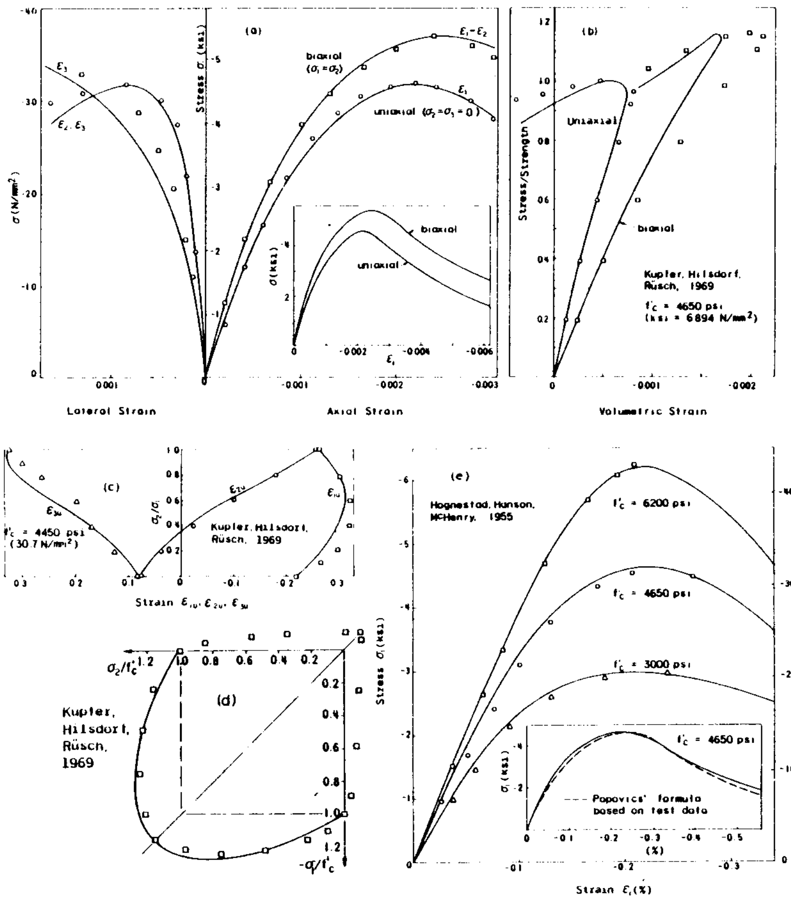


**FIG. 3.—Fits of Uniaxial and Biaxial Test Data: *(a)* Axial and Lateral Strains; *(b)* Volume Change; *(c)* Failure Strains; *(d)* Failure Envelope; *(e)* Uniaxial Tests**

Drucker's postulate [$\Delta W \geq 0$, Fig. 1(*o*); Eq. 2] is satisfied for an unload-reload cycle if $c_u < c_r < 2c_u$, and this is fulfilled by the values of $c_1$ identified by data fitting $c_u = 0.5$ and $c_r = 0.8$.

The fits of typical test data for unloading and cyclic load (23,38,39,40) are shown in Fig. 2.

Alternatively, we could have used as "associated" loading-unloading criteria the conditions that $d\mu$ or $d\kappa$ would change their signs.



**Remarks on Stiffness Parallel to Loading Surface**

The present model gives zero plastic (as well as fracturing) strain increment if the loading proceeds in the direction parallel to the current loading surface. Although experimental evidence is lacking, this response is no doubt too stiff (8). Some plastic strain should be produced by this loading, too. The endochronic

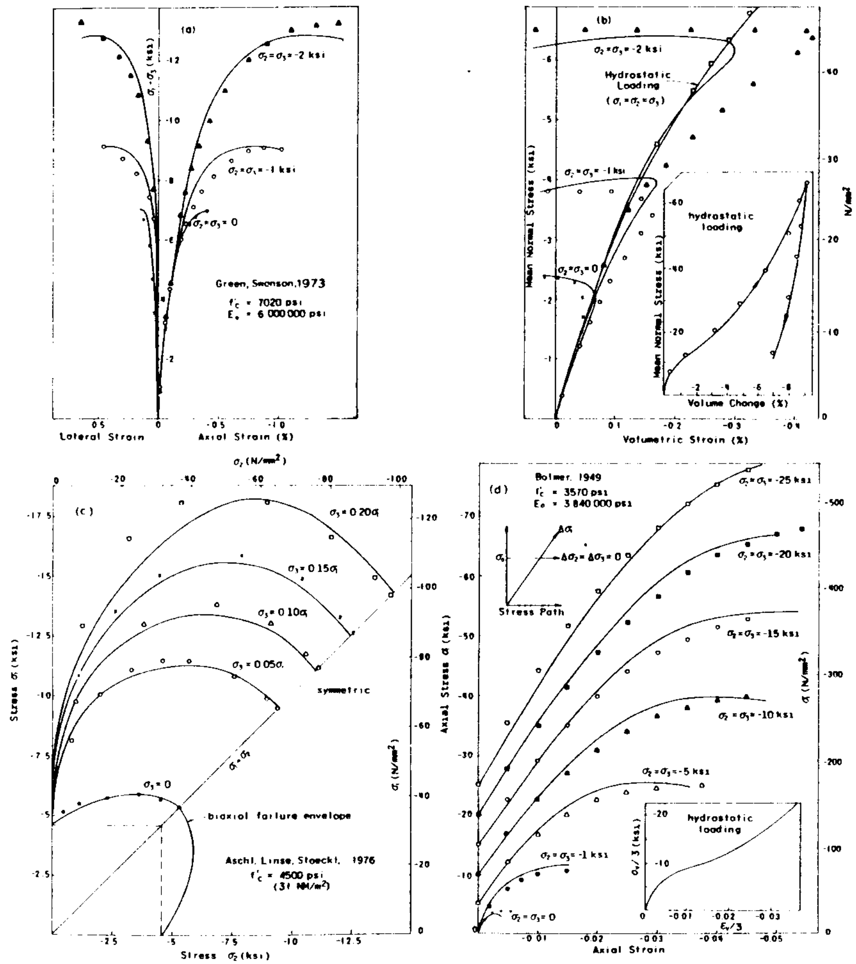


**FIG. 4.—Fits of Triaxial Test Data: *(a,b,d)* Standard Triaxial Tests; *(b)* Volume Change; *(c)* Failure Envelopes for Proportional Triaxial Loading**

theory does give nonzero inelastic strain increments for these loading directions, thus predicting a softer response. [This is manifested by the fact that for the present theory the inelastic stiffness locus (8) is a straight line while for the endochronic theories it is a circle or a quadratic curve.]

A simple way to achieve a softer response for loading that is parallel to the deviatoric loading surface yet at the same time cause no effect on loading that is normal to the loading surface is to insert at the right-hand side of Eq. 20 the term

$$-c_0 \frac{\phi}{2}\left(de_{ij} - e^*_{ij}\frac{d\bar{\gamma}^*}{\bar{\gamma}^*}\right), \quad \text{with} \quad d\bar{\gamma}^* = \frac{e^*_{km}\, d\epsilon_{km}}{2\bar{\gamma}^*} \qquad (28)$$

with some coefficient $c_0$. This term vanishes for proportional loading, as can be checked by substituting $de_{ij} = e^*_{ij}\,ds$, $de_{km} = e^*_{km}\,ds$ and is also zero for hydrostatic loading. Thus, the effect on the fits in Figs. 3–5 is small. For loading that substantially deviates from the $e^*_{ij}$ direction, this term becomes significant, making the response softer. Geometrically, this term can be interpreted as a

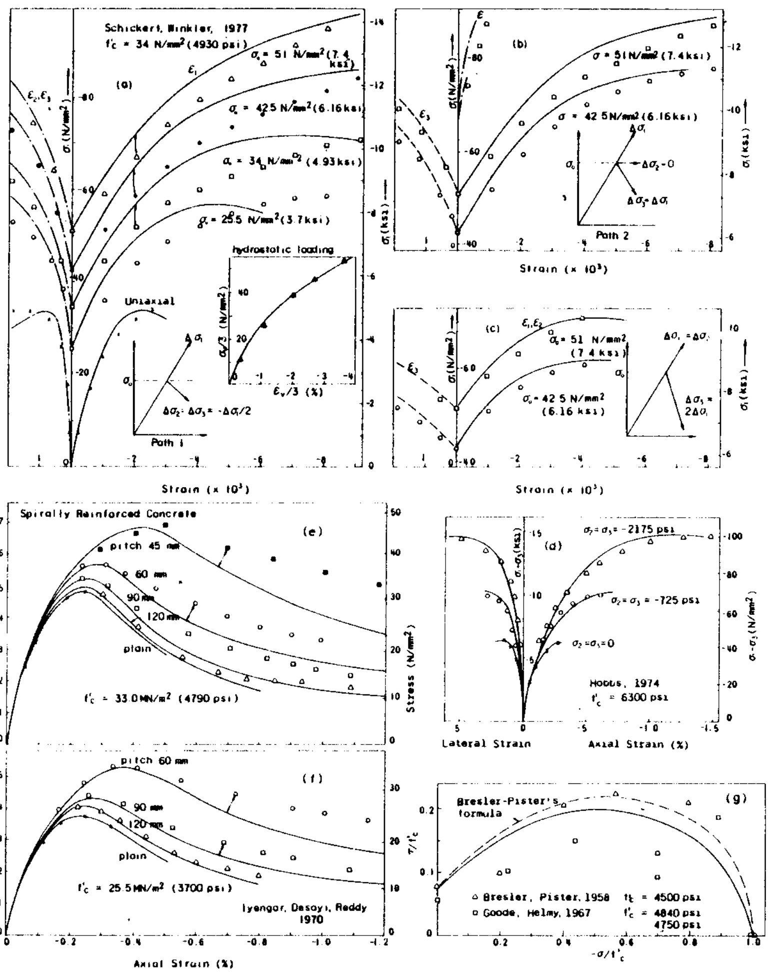


FIG. 5.—Fits of: (*a,b,c*) Nonproportional Triaxial Tests; (*d*) Standard Triaxial Tests; (*e, f*) Tests of Spirally Reinforced Cylinders; (*g*) Shear-Compression Tests



sort of vertex hardening (8), and it is interesting to note that it makes the response closer to the endochronic theory (8), i.e., the inelastic stiffness locus (8) becomes curved. A similar term, but in the stress rather than strain space, has been introduced by Rice and Rudnicki (36); however, in the case of strain softening in which $d\bar{\gamma}^*$ is positive while $d\bar{\tau}^*$ is negative, Eq. 28 is more appropriate.

Experimental determination of coefficient $c_0$ in Eq. 28 is hardly possible at present because there exist no good and comprehensive data on nonproportional loading paths for concrete. Until they become available, it seems suitable to consider $c_0 = 1$, which gives about the same stiffness for the parallel strain increments as for the normal strain increments and gives results rather close to endochronic theory.

## Summary and Conclusions

Incremental plasticity and fracturing (microcracking) material theory are combined to obtain a nonlinear triaxial constitutive relation that is incrementally linear. A new hardening rule, called jump-kinematic hardening, is used for unloading, reloading, and cyclic loading. In the case of continuous tensile cracks, the theory applies only for the solid (albeit microcracked) concrete between the cracks. Because of friction and dilatancy due to shear, the tangential moduli are nonsymmetric.

The theory combines the plastic stress decrements with the fracturing stress decrements, which reflect microcracking, and accounts for internal friction, pressure sensitivity, inelastic dilatancy due to microcracking, strainsoftening, degradation of elastic moduli due to microcracking, and the hydrostatic nonlinearity due to pore collapse. Failure envelopes are obtained from the constitutive law as a collection of the peak points of the stress-strain response curves. The jump-kinematic hardening allows for inelastic response during unloading, reloading, and cyclic loading and, at the same time, it does not in itself cause violation of Drucker's postulate. As a consequence of the incremental linearity, the plastic strain increments vanish for loading that is parallel to the loading surface; this response may be too stiff and questionable for material instability predictions.

Six scalar material functions are needed to fully define the monotonic response. One function, the dilatancy due to microcracking, is determined theoretically, based on Budianski and O'Connell's calculation of the effective elastic constants of a randomly microcracked elastic material by the self-consistent method for composites.

All material parameters are identified by fitting published test data, and their dependence on concrete strength is also given. The fits are as good as those for the previous endochronic theory, except for a high number of cycles and the time dependence. The theory is ready for use in finite element programs.

## Acknowledgment

Grateful appreciation is due to the National Science Foundation for supporting this research under Grants ENG 75-14848 and ENG 75-14848-A01 to Northwestern University. Part of the work was also supported under an award of the Guggenheim Fellowship to the first writer.

## Appendix I.—Material Functions and Data Fits

**Degradation of Elastic Moduli.**—According to Eqs. 36, 44″, and 42″ of Ref. 14

$$f_K(\nu) = 1 - \frac{16}{9}\left(\frac{1-\nu^2}{1-2\nu}\right)c; \quad f_G(\nu) = 1 - \frac{8}{45}(10-7\nu)c,$$

$$\text{with} \quad c = \frac{45}{8}\,\frac{\nu-\nu_0}{(1+\nu)(10\nu_0 - \nu - 8\nu_0\nu)} \quad \ldots\ldots\ldots\ldots \quad (29)$$

in which $\nu_0$ = initial Poisson ratio. These expressions are precisely those obtained for long elliptical cracks (14). The expressions for circular cracks may be more pertinent but the crack shape does not affect $f_K$ by more than a few percent, thus, the expressions for long cracks are preferred over those for circular cracks because they are simpler (14).

**Inelastic Material Functions.**—These have been identified as follows:

$$h = \frac{a_0 - a_1\bar\tau + |I_3 f'_c|^{1/4}(a_2 + a_3 J_{31}) + a_4 I_3}{J_2 - a_5 J_{31}};$$

$$\beta' = \frac{\bar\tau}{f'_c + I_1 - a_6\bar\tau} \quad \ldots\ldots\ldots\ldots \quad (30)$$

$$\phi = G\bar\gamma\,\frac{b_0 + b_1 J_2 + b_2 J_{31}}{(f'_c + b_3 I_1)^2 + b_4 J_{31} + I_3(b_5 + b_6\bar\gamma)};$$

$$\alpha' = \alpha_0 + \frac{\alpha_1 I_3 - \alpha_2 I_3^{2/3}}{\alpha_3 + I_3^{2/3}(\alpha_4 + \alpha_5 J_2)} \quad \ldots\ldots\ldots\ldots \quad (31)$$

$$\beta = \left(\frac{\beta'' J_2^\epsilon}{1 + \beta'' J_2^\epsilon}\right)^2; \quad \beta'' = \frac{c_0 J_2 + c_1 J_{31}}{(c_2 + c_3 I_1)^2 + (c_4 I_3 - c_5 J_3)(I_1)^{-1}} \quad \ldots\ldots \quad (32)$$

in which $J_{31} = |J_3/I_1|$; $I_1, I_2, I_3$ = first, second, and third invariants of $\sigma^*_{ij}$; $J_2, J_3$ = second and third invariant of stress deviator $s^*_{ij}$; $J_2^\epsilon$ = second invariant of strain deviator $e^*_{ij}$; $I_1, I_2, I_3$ are always taken in absolute value if negative, $I_1 = |3\sigma^*| = |\sigma^*_{kk}|$, $I_3 = |\det(\sigma^*_{ij})|$; $J_2 = \bar\tau^{*2} = 1/2\ s^*_{km}\,s^*_{km}$; $J_3 = I_3 - \sigma J_2 + \sigma^3$; $J_2^\epsilon = \bar\gamma^{*2} = 1/2\ e^*_{km}\,e^*_{km}$; and

$$a_0 = \frac{f_c'^4}{90\text{ psi}}; \quad a_1 = \frac{f_3'^3}{150\text{ psi}}; \quad a_2 = \left(\frac{15\times10^7\text{ psi}}{f'_c}\right)^2;$$

$$a_3 = \left(\frac{35{,}000\text{ psi}}{f'_c}\right)^3; \quad a_4 = \left(\frac{f'_c}{2{,}700\text{ psi}}\right)^2; \quad a_5 = 1.95,\ a_6 = 1.73 \quad \ldots\ldots \quad (33a)$$

$$b_0 = (2\text{ psi})^2; \quad b_1 = 36{,}000; \quad b_2 = 13\times10^4; \quad b_3 = \frac{14{,}000\text{ psi}}{f'_c};$$

$$b_4 = 134; \quad b_5 = \left(\frac{45{,}000\text{ psi}}{f'_c}\right)^{2.5}\text{psi}; \quad b_6 = \left(\frac{f'_c}{840\text{ psi}}\right)^5 \quad \ldots\ldots\ldots \quad (33b)$$



$$c_0 = 96\times10^5; \quad c_1 = 405\times10^5; \quad c_2 = 4{,}650\text{ psi}; \quad c_3 = \frac{14{,}000\text{ psi}}{f'_c};$$

$$c_4 = \left(\frac{f'_c}{1{,}350\text{ psi}}\right)^6; \quad c_5 = 110 \quad \ldots\ldots\ldots\ldots \quad (33c)$$

$$\alpha_0 = 0.5; \quad \alpha_1 = \frac{2{,}500\text{ psi}}{f_c'^2}; \quad \alpha_2 = \left(\frac{11{,}000\text{ psi}}{f'_c}\right)^{0.8}; \quad \alpha_3 = (1\text{ psi})^2;$$

$$\alpha_4 = \left(\frac{2{,}100\text{ psi}}{f'_c}\right)^{1.6}\times10^{-6}; \quad \alpha_5 = \frac{f'_c}{(5{,}540\text{ psi})^4} \quad \ldots\ldots\ldots\ldots \quad (33d)$$

in which psi = 6.895 kN/m$^2$. The initial elastic moduli are $G_0 = E_0/2(1+\nu_0)$, $K_0 = E_0/3(1-2\nu_0)$ in which $\nu_0 = 0.18$; and $E_0$ is determined by an adjusted American Concrete Institute (ACI) expression:

$$E_0 = (0.9\text{ psi} + 0.00006 f'_c)(57{,}000\sqrt{f'_c})(\text{psi})^{-1/2} \quad \ldots\ldots\ldots\ldots \quad (34)$$

**Analysis of Material Functions.**—They are basically introduced in the following form:

$$h = \frac{a_0 + h_b(I_3)}{h_a(\bar\tau)}; \quad \phi = \frac{b_0 + \phi_b(\bar\tau)}{\phi_a(\sigma) + \phi_c(I_3)}\bar\gamma; \quad \beta = \frac{\beta_d(\bar\tau)\,\beta_e(\bar\gamma)}{\beta_f(\sigma)} \quad \ldots\ldots \quad (35)$$

$$\beta' = \frac{\beta_a(\bar\tau)}{\beta_b(\sigma) - \beta_c(\bar\tau)}; \quad \alpha' = \alpha_0 + \frac{\alpha_a(I_3)}{\alpha_3 + \alpha_b(I_3)\,\alpha_c(\bar\tau)} \quad \ldots\ldots\ldots\ldots \quad (36)$$

The plastic hardening modulus, $h$, must be infinite for hydrostatic compression since there is no plastic volume change, and it must also tend to infinity at small stress [Figs. 1(*g*) and 1(*j*)], since plastic strains at small stresses are negligible. This indicates that the denominator of the expression for $h$ must depend on $\bar\tau$ and must vanish as $\bar\tau \to 0$. (The $\bar\gamma$ should not be used because plasticity declines as the strain softening advances.) The confining pressure restricts plastic strain, but mainly in triaxial loading and not much in biaxial loading. This suggests that function $h_b$ should decrease with $I_3$ as the triaxial compression test progresses ($I_3 = |\sigma_1\sigma_2\sigma_3|$).

The fracturing compliance, $\phi$, must increase if the shear strain increases, which can be modeled by an increasing function, $\phi_b(\bar\tau)$ [Fig. 1(*g*)]. Furthermore, $\phi$ must vanish at hydrostatic compression loading, for the nonlinearity of the hydrostatic diagram stems from fracturing. Thus, since $d\kappa$ in Eq. 20 is divided by $\bar\gamma$, then $\phi$ must be proportional to $\bar\gamma$ to counteract it (Eq. 35). On the other hand, fracturing must become rather limited at high confining pressure, so Eq. 35 for $\phi$ must be divided by an increasing function of $|\sigma|$, i.e., $\phi_b(\sigma)$. Fracturing must get intensified as shear stress increases, so $\phi$ must increase with $\bar\tau/\sigma$. Furthermore, fracturing must be much more resitricted under triaxial compression than under uniaxial compression. This can be introduced in Eq. 35 by function $\phi_c(I_3)$, which increases in absolute value.

The volumetric fracturing compliance coefficient, $\alpha'$, adjusts $\phi$ so as to represent the volumetric nonlinearity, chiefly due to hydrostatic compression, $-\sigma$ [Fig. 1(*k*)]. This effect is much more significant in hydrostatic compression tests than in biaxial or uniaxial tests. Thus, it should be governed by $I_3$ rather than $(-\sigma)$. The hydrostatic compression test first indicates a softening, due to pore collapse [Fig. 1(*k*)], which is indicated by function $\alpha_a$ growing from zero; however, later there is a stiffening again after the pores have closed, and this is modeled by an increasing function $\alpha_b(I_3)$ in the denominator of $\alpha'$, Eq. 36. The last effect, however, should be delayed or eliminated by simultaneous shear; therefore, an increasing function, $\alpha_c(\bar{\tau})$, should multiply $\alpha_b(I_3)$ in Eq. 36.

The plastic friction coefficient, $\beta'$, may be approximately interpreted as the tangent slope of Mohr's failure envelope [Fig. 1(*j*)]. This slope decreases as the hydrostatic pressure, $-\sigma$, increases, so the expression for $\beta'$ (Eq. 36) may be divided by an increasing function, $\beta_b(\sigma)$. A lower friction coefficient means a more plastic response, and the response is most plastic when $-\sigma$ is high and shear $\bar{\tau}$ is low (triaxial test). So, $\beta'$ should increase with $\bar{\tau}$, which is introduced by an increasing function, $\beta_a(\bar{\tau})$.

The plastic dilatancy factor, $\beta$, must be always non-negative to give dilatancy rather than densification. The dilatancy is high at high shear stresses or shear strains and low at high confining pressures. Thus, $\beta$ must increase with $\bar{\tau}$ and $\bar{\gamma}$ [Fig. 1(*l*)], and it must decrease with $-\sigma$, which is modeled by increasing functions $\beta_d(\bar{\tau})$, $\beta_e(\bar{\gamma})$, and $\beta_f(\sigma)$.

Compared to the basic form of the expressions for $h$, $\phi$, $\alpha'$, $\beta'$, and $\beta$ deduced before, various further terms were introduced to "tune up" the fits empirically. For this purpose the desired trends of various functions were sketched as in Figs. 1(*g*), 1(*h*), 1(*i*), 1(*j*), and 1(*k*). The interrelation of the fits of uniaxial, biaxial, and triaxial tests was controlled by noting that $I_1$ is nonzero for all three, $J_2$ is nonzero only for biaxial and triaxial tests, and $I_3$ is nonzero only for triaxial tests. The ratio $J_3/I_1$ appeared to be useful to control the inelastic dilatancy since it grows with triaxial compression and decreases with shear stress. The invariant $J_3$ may also be used to model the deviation from the hydrostatic stress influence.

**Simulation of Test Conditions.**—The incremental equation, Eq. 21, may be rewritten as $d\boldsymbol{\sigma} = \mathbf{C}\, d\boldsymbol{\epsilon}$ in which $\boldsymbol{\sigma}$, $\boldsymbol{\epsilon}$ = (6 × 1) column matrices of stress and strain components, i.e., $\boldsymbol{\sigma} = (\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{31})^T = (\sigma_1, \sigma_2, \ldots, \sigma_6)^T$, in which $T$ denotes a transpose; and $\mathbf{C}$ = (6 × 6) matrix of tangential moduli (nonsymmetric). To integrate this stress-strain relation for a particular type of test, some stress components and some other strain components are prescribed. To exemplify it, consider an ideal biaxial test, for which $\Delta\epsilon_{11}$ and $\Delta\epsilon_{22}$ are specified for each load step and $\Delta\sigma_{33} = \Delta\sigma_{12} = \Delta\sigma_{23} = \Delta\sigma_{31} = 0$. Denoting the column submatrices of the known strain and stress increments as $\Delta\boldsymbol{\epsilon}^k = (\Delta\epsilon_{11}, \Delta\epsilon_{22})^T$ and $\Delta\boldsymbol{\sigma}^k = (\Delta\sigma_{33}, \Delta\sigma_{12}, \Delta\sigma_{23}, \Delta\sigma_{31})^T$ and the unknown ones as $\Delta\boldsymbol{\epsilon}^u = (\Delta\epsilon_{33}, \Delta\epsilon_{12}, \Delta\epsilon_{23}, \Delta\epsilon_{31})^T$ and $\Delta\boldsymbol{\sigma}^u = (\Delta\sigma_{11}, \Delta\sigma_{22})^T$, we may write the incremental relations in the partitioned form

$$\begin{Bmatrix} \Delta\boldsymbol{\sigma}^u \\ \Delta\boldsymbol{\sigma}^k \end{Bmatrix} = \begin{bmatrix} \mathbf{P}, & \mathbf{Q} \\ \mathbf{R}, & \mathbf{S} \end{bmatrix} \begin{Bmatrix} \Delta\boldsymbol{\epsilon}^k \\ \Delta\boldsymbol{\epsilon}^u \end{Bmatrix} \quad \text{(37)}$$

The solution of $\Delta\boldsymbol{\sigma}^u$ and $\Delta\boldsymbol{\epsilon}^u$ is then obtained as

$$\begin{Bmatrix} \Delta\boldsymbol{\sigma}^u \\ \Delta\boldsymbol{\epsilon}^u \end{Bmatrix} = \begin{bmatrix} \mathbf{P} - \mathbf{Q}\mathbf{S}^{-1}\mathbf{R}, & \mathbf{Q}\mathbf{S}^{-1} \\ -\mathbf{S}^{-1}\mathbf{R}, & \mathbf{S}^{-1} \end{bmatrix} \begin{Bmatrix} \Delta\boldsymbol{\epsilon}^k \\ \Delta\boldsymbol{\sigma}^k \end{Bmatrix} \quad \text{(38)}$$



An equation of this type is automatically set up in the program (in component form) for various specified types of tests to be simulated and fitted.

**Basic Information on Test Data Used.**—This has been summarized in Ref. 9, except for the following.

*Fig. 4(a, b).*—2.7-in. × 6-in. cylinders; age: over 200 days; strain-controlled test; strain rate: $10^{-4}$/sec; water-cement-aggregate ratio: 0.6:1:6.3 by weight; cured for 4 days at constant environment (curing room); covered with wet burlap, then exposed to drying lab environment for several days until testing.

*Fig. 4(c).*—10-cm cubes; cured for 1 week in wet room, then sealed and stored at 18° C and 65% relative humidity for 20 days until testing; age: 3 months–4 months at test; water-cement-sand-gravel: ratio 0.81:1:3.65:3.19; stress-controlled test: specimens glued on brush bearing platens.

*Fig. 5(a–c).*—10-cm cubes; demolded after 1 day; cured for 1 week in 100% relative humidity, then sealed in plastic bags; age: 150 days–250 days at test; water-cement-sand-gravel ratio: 0.84:1:3.16:3.2 by weight; stress-controlled test with flexible steel platens; loading rate 0.075 N/mm$^2$/s.

*Fig. 5(d).*—Specimens cured in a fog room for at least 7 days, then sealed in plastic bags for 7 days until testing; maximum aggregate size: 4.76 mm.

*Fig. 2(a, b).*—250-mm$^3$ × 75-mm$^3$ × 75-mm$^3$ prisms; strain-controlled test; strain rate: $2 \times 10^{-3}$/min; epoxy between specimen face and platen.

*Fig. 2(e–g).*—10-cm$^3$ × 10-cm$^3$ × 50-cm$^3$ prisms; water-cement-aggregate ratios: (1) 0.7:1:5.7, (2) 0.99:1:7, and (3) 0.58:1:5.1; demolded after 1 day, in water at 18° C–20° C for 7 days, then stored at 55%–70% relative humidity and 15° C–19° C until test; age: 56 days at test.

## Appendix II.—References


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<u>Errata</u>

Page 413, 1-st line after Eq. 16: Should read "$\bar{Y}^*$" instead of "$\bar{Y}$" and "$\epsilon^*$" instead of "$\epsilon$ ".

Page 417, Eq. 23: Should read "$dW' = 3\sigma \, d\epsilon^{el}$, $d\epsilon^{el} = d\sigma/3K$, $dW = s_{km} de_{km}$"

Page 417, Line 4 and Eq. 23: Symbols "W" and "W' " should be mutually interchanged.

Page 417, 1-st line after Eq. 24f: Should read "$\dot{W}_o$" instead of "W".

Page 417, 1-st and 2-nd line after Eq. 26: Symbols "$W_o$" and "$W_o'$" should be interchanged.

Page 423, Eq. 29: Should read "$\nu_o - \nu$" instead of "$\nu - \nu_o$".

Page 423, Eq. 33b: Should read "$b_5 = (45{,}000 \text{ psi}/f_c')^{2.5}/\text{psi}$" and "$b_6 = (f_c'/840 \text{ psi})^5/\text{psi}$" instead of "$b_5 = (45{,}000 \text{ psi}/f_c')^{2.5} \text{psi}$ and "$b_6 = (f_c'/840 \text{ psi})^5$".

Page 423, lines 4 and 5 after Eq. 32: Should read "$J_3 = I_3 + \sigma J_2 - \sigma^3$" instead of "$J_3 = I_3 - \sigma J_2 + \sigma^3$".

Page 423, Eq. 33a: Should read "$a_2 = [15 \times 10^7 (\text{psi})^2/f_c']^2$" instead of "$a_2 = (15 \times 10^7 \text{psi}/f_c')^2$", and "$f_c'^3$" instead of "$f_3'^3$."

Page 423, line after Eq. 32: Should read "$J_{31} = J_3/I_1$" instead of "$J_{31} = |J_3/I_1|$".

Page 424, Eq. 33d: Should read "$\alpha_5 = f_c' (5{,}540)^{-4} (\text{psi})^{-3}$" instead of "$\alpha_5 = f_c' (5{,}540 \text{ psi})^{-4}$"

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**14653 PLASTIC-FRACTURING THEORY FOR CONCRETE**

KEY WORDS: **Concrete; Concrete structures;** Constitutive equations; Cracking; Ductility; **Failure; Fracturing;** Inelastic action; Mathematical models; **Plasticity;** Tests

ABSTRACT: Incremental plasticity and fracturing (microcracking) material theory are combined to obtain a nonlinear triaxial constitutive relation that is incrementally linear. The theory combines the plastic stress decrements with the fracturing stress decrements, which reflect microcracking, and accounts for internal friction, pressure sensitivity, inelastic dilatancy due to microcracking, strain softening, degradation of elastic moduli due to microcracking, and the hydrostatic nonlinearity due to pore collapse. Failure envelopes are obtained from the constitutive law as a collection of the peak points of the stress strain response curves. Six scalar material functions are needed to fully define the monotonic response. One function, the dilatancy due to microcracking, is determined theoretically based on Budianski-O'Connell's calculation of the effective elastic contents of a randomly microcracked elastic material by the self consistent method for composites.

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