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## ENDOCHRONIC CONSTITUTIVE LAW FOR LIQUEFACTION OF SAND

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### INTRODUCTION

Mathematical analyses involving the liquefaction of sand deposits require a realistic constitutive law for the nonlinear behavior of sand, especially under conditions of cyclic shear. Although considerable effort has been devoted to the development of constitutive laws based on theories of plasticity and hypoelasticity (7,9) the success of these theories has been quite limited, and no constitutive law is presently available to adequately describe the densification and hysteresis of sand subjected to cyclic shear. Practical studies have led to the establishment of simplified empirical stress-strain relationships in which the material properties (such as shear modulus, damping coefficient, densification parameter, etc.) depend on the number of cycles. However, this variable may not appear in any general constitutive law that is to be applied to other than sinusoidal load histories, such as those typical of earthquakes. Also, these formulations are not in a form that would be applicable to multidimensional situations and general drainage conditions.

Developed herein is an entirely new type of constitutive law that achieves the aforementioned objectives. This law may be termed "endochronic" because the nonlinearity of the material is introduced chiefly by means of an intrinsic time parameter, which is an independent scalar variable whose increments depend generally on both time increments and deformation increments. The dependence on time increments disappears, of course, in the case of time-independent inelastic behavior, as is the case with sand. In this case the intrinsic time parameter represents the length of the path traced by the states of the material in a six-dimensional space (with a general metric tensor) whose coordinates are the strain components. The length of this path constitutes a natural choice for the

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independent variable, and it has been introduced in many theoretical works (cf. 3). However, the usefulness of this path parameter for the practical description of inelastic behavior was apparently not realized until Valanis (13) employed it for metals to develop a stress-strain law that correctly characterizes strain-hardening, unloading behavior, the effect of pretwist on axial behavior, the contraction of hysteresis loops with cyclic straining, and the strain rate effect. From the work of Valanis it became apparent that endochronic theory provides a very fruitful approach to describing materials in which inelastic strain develops gradually.

Using Valanis' theory as a point of departure, a method of extending endochronic theory to concrete, rock, and sand has been proposed in general terms in Ref. 1, and the detailed formulation for concrete has been given in Ref. 3 (to appear in this Journal). This extension has required the introduction of several novel concepts to handle: (1) The sensitivity of intrinsic time to hydrostatic pressure; (2) inelastic dilatancy (or densification) due to shear straining; and (3) the strain-softening tendency. The purpose of this work is to present the detailed development of endochronic theory for sand and to show how this theory can be used in the analysis of liquefaction. As developed herein, the law is particularly applicable to low stress oscillatory loadings at low pressures that occur during earthquakes, but extensions might be formulated to handle strains at high pressures associated with blast loads.

### REARRANGEMENT MEASURE

The source of inelasticity in sand is the irreversible rearrangement of grain configurations associated with deviatoric strains. Thus, it is convenient to characterize the accumulation of rearrangement by an appropriate variable,  $\xi$ , termed the rearrangement measure, which will be used as the independent variable in the stress-strain law. For this reason and also to ensure that the energy dissipation rate is always positive,  $\xi$  may never decrease. Since the rearrangement of grain configurations results from deformation,  $d\xi$  is a function of the strain increments,  $d\epsilon_{ij}$ . Assuming that the development of inelastic strain is gradual,  $d\xi$  must be a continuous and smooth function of  $d\epsilon_{ij}$ , and  $(d\xi)^s$  with a certain exponent may be expanded in a tensorial power series as

$$(d\xi)^s = p^{(0)} + p_{ij}^{(1)} d\epsilon_{ij} + p_{ijkl}^{(2)} d\epsilon_{ij} d\epsilon_{kl} + p_{ijklmn}^{(3)} d\epsilon_{ij} d\epsilon_{kl} d\epsilon_{mn} + \dots \quad (1)$$

in which latin indices refer to cartesian coordinates  $x_i$  ( $i = 1, 2, 3$ ); and repeated indices imply summation from one to three. Since  $d\xi$  must vanish as  $d\xi_{ij} \rightarrow 0$ , it is necessary that  $p(0) = 0$ . Furthermore, the condition that the energy dissipation rate must be always positive requires that  $p_{ij}^{(1)} = 0$  and that the coefficients,  $p_{ijkl}^{(2)}$ , define a positive definite quadratic form. Because  $d\epsilon_{ij}$  is infinitesimal, terms of order higher than two are negligible; accordingly,  $p_{ijklmn}^{(3)} = 0$ , etc. Since the right-hand side of Eq. 1 is second-order small,  $s$  must equal two. Finally, it is necessary to meet the condition of isotropy and the condition that  $d\xi = 0$  for purely volumetric strains, because no rearrangement of grain configurations takes place in this case. By use of the same reasoning as that applied in developing endochronic theory for concrete (see Ref. 3, Eqs. 2-5), it can be shown that the only expression satisfying the latter two conditions

$$d\xi = \sqrt{J_2(d\epsilon)} = \sqrt{\frac{1}{2} de_{ij} de_{ij}} \dots \dots \dots (2)$$

in which  $J_2(\ )$  = the second invariant of the deviator of the tensor in parentheses;  $\epsilon = [\epsilon_{ij}]$  = the strain tensor;  $e_{ij} = \epsilon_{ij} - \delta_{ij} \epsilon_{kk}/3$  = the deviator of the tensor,  $\epsilon_{ij}$ ; and  $\delta_{ij}$  designates Kronecker delta. A multiplicative constant is omitted in Eq. 2 because  $d\xi$  will be multiplied in the stress-strain law by an arbitrary constant. In the special case of pure shear strain,  $\gamma = 2 \epsilon_{12}$  and Eq. 2 yields

$$d\xi = \frac{1}{8} |d\gamma| \dots \dots \dots (3)$$

More specifically, for the case of cyclic pure shear in which  $\gamma = \gamma_o \sin \omega t$ , Eq. 3 yields  $\Delta\xi = \gamma_o/2$  per cycle, so that at the end of the  $N$ th cycle  $\xi$  may be expressed as

$$\xi = \frac{1}{2} \gamma_o N = \frac{1}{4} \gamma_o \frac{\omega}{\pi} t \dots \dots \dots (4)$$

Thus,  $\xi$  is a measure of the number of cycles in the loading history.

**DENSIFICATION DUE TO DEVIATORIC STRAINS**

The densification of sand may be characterized by the volumetric strain,  $\lambda$ , whose sign is chosen to be negative when the volume decreases. If vertical accelerations are not large enough to cause any significant separation or jumping of particles (as is the case in many earthquakes), the densification is produced almost exclusively by interparticle slips that result in a rearrangement of grain configurations. Subject to this restriction,  $d\lambda$  must be proportional to  $d\xi$ , and the dependence of the densification increment per cycle of shear on the strain amplitude and the number of cycles may be expressed by

$$d\lambda = - \frac{d\kappa}{c(\kappa)}; \quad d\kappa = C(\epsilon, \sigma) d\xi \dots \dots \dots (5)$$

The function  $c(\kappa)$  models the decrease in the densification increments per cycle with an increase in the number of cycles,  $N$ . Accordingly,  $c$  is a continuous monotonically increasing positive function, which may be termed the densification-hardening function. Eq. 5 may be integrated to yield  $\lambda = \lambda(\kappa)$  and inverted to give  $\kappa = \kappa(\lambda)$ . Because  $c(\kappa)$  is always positive, the simplest choice for  $c$  is the linear expression

$$c(\kappa) = c_o (1 + \alpha \kappa) \dots \dots \dots (6)$$

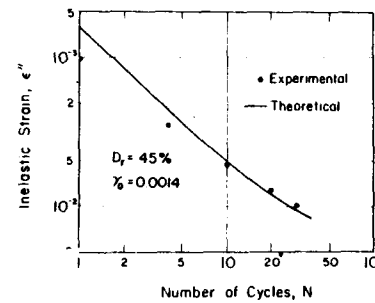
for which  $\lambda = - \frac{1}{c_o \alpha} \ln(1 + \alpha \kappa); \quad c(\kappa) = c_o e^{-c_o \alpha \lambda} \dots \dots \dots (7)$

in which  $\alpha$  and  $c_o$  are constants for a given sand at a given relative density. Integration is also easy for the more general function,  $c \sim (1 + \alpha \kappa/k)^k$ . The function,  $C$ , may generally introduce a dependence of densification rate on volumetric stress  $\sigma$ ; however, for the sand studied by Silver and Seed (12) no dependence on  $\sigma$  was discerned, and thus  $C$  may be considered independent

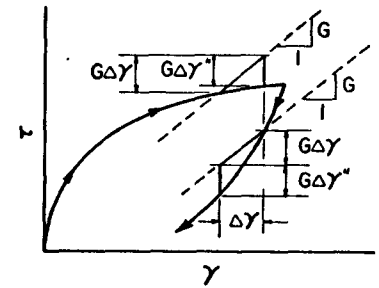
of  $\sigma$  (at least for some sands). The function,  $C(\epsilon, \sigma)$ , which may be called the densification-softening function, models the increase in densification rate with increasing strain amplitude,  $\gamma_o$ , in cyclic shear. Because of isotropy, only the invariants of the strain tensor,  $\epsilon$ , may appear in  $C$ . The third invariant,  $I_3(\epsilon)$ , may be neglected because it is third-order small when strains are small; however, even if it were not neglected for this reason,  $I_3(\epsilon) = 0$  in the case of pure shear. The first invariant,  $I_1(\epsilon)$ , need not be considered because the dependence on  $I_1(\epsilon)$  is equivalent to the dependence on relative density  $D_r$ , which will be considered separately. Thus,  $C$  may be regarded as a function of only  $J_2(\epsilon)$ , the second invariant of the deviator of  $\epsilon$ , and one suitable choice for  $C$  is

$$C(\epsilon) = 2q [8J_2(\epsilon)]^{(q-1)/2} \dots \dots \dots (8)$$

in which  $q$  is a non-negative constant for a given sand. In the special case of pure shear, Eq. 8 becomes  $C(\epsilon) = 2q |\gamma|^{q-1}$  and Eq. 5 yields  $d\kappa = (1/4)q |\gamma|^{q-1} |d\gamma|$ . For cyclic shear the increment of  $\kappa$  over the cycle is  $4[(1/4)q \gamma_o^q]$ ,



**FIG. 1.—Densification versus Number of Cycles**



**FIG. 2.—Elastic and Inelastic Stress Increments**

in which  $\gamma_o$  = the amplitude of shear strain. Thus, after  $N$  cycles  $\kappa = \gamma_o^q N$ , and Eq. 7 becomes

$$\gamma = - \frac{1}{c_o \alpha} \ln(1 + \alpha \gamma_o^q N) \dots \dots \dots (9)$$

To illustrate the ability of this theory to describe experimental results, Eq. 9 has been fitted to test data reported by Seed and Silver (12) for the densification of a given sand at a relative density,  $D_r$ , of 45% under a cyclic shear strain of 0.0014, and a typical fit is shown in Fig. 1 (for the case illustrated, the values of the material parameters are  $q = 1.4$ ,  $\alpha = 1.0$ , and  $c_o = 1.0$ ). The material parameters,  $\alpha$ ,  $q$ , and  $c_o$ , will generally depend on the relative density,  $D_r$ :

$$\alpha = \alpha(D_r); \quad q = q(D_r); \quad c_o = c_o(D_r) \dots \dots \dots (10)$$

Although the nature of this dependence requires further investigation, it may be concluded even without quantitative evaluation of test data, that  $c_o(D_r)$  must approach zero as  $D_r$  approaches a certain critical density,  $D_r^{cr}$ , and it must become negative when  $D_r > D_r^{cr}$  to reflect the fact that shear at overcritical

densities causes an increase in volume (called dilatancy) rather than densification. One suitable function may be

$$c_o = \pm C_o \left| 1 - \frac{D_r}{D_r^c} \right|^r \dots \dots \dots (11)$$

in which  $C_o$  and  $r$  are constants for all densities of a given sand, but may perhaps depend on confining stress. The critical density,  $D_r^c$ , for a given sand will also depend on the confining stress and possibly other variables [e.g.,  $J_2(\epsilon)$ ]. For  $D_r < D_r^c$ , the relative density gradually increases due to shear straining.

Because the dependence of  $D_r$  on  $\lambda$  would considerably complicate the analysis,  $D_r$  must be approximated by the constant,  $D_r^o$ . In addition, this densification law will probably not be too accurate for describing the detailed progress of densification within a given cycle. For example, the application of a small increment of shear strain within a cycle may cause a dense sand to dilate, even though the repeated application of many cycles of shear strain may produce a negative cumulative volume change or densification. This phenomenon cannot be made by this theory unless  $c_o$  is considered to be a function of  $\epsilon$ . However, for most practical applications the detailed progress of densification within a given cycle is not of major concern. Generally, the dependence of  $D_r$  on  $\lambda$  is not too important, because the change in  $D_r$  that is normally produced by shear straining alone usually does not exceed about 5% (based on the assumptions that the relative density range from 0%-100% corresponds to approximately a 15% change in the actual density of sand and that shear straining alone generally does not cause volumetric strains of more than about 1%). The foregoing formulation of the densification law is analogous to that employed in characterizing the dilatancy of concrete (1,3); the only difference consists in the form of the functions,  $C(\epsilon)$  and  $c(\kappa)$ . However, with regard to metals (13) the inclusion of dilatancy or densification represents one of the major extensions needed.

**INTRINSIC TIME**

Since the increments of irreversible (inelastic) strains in sand are caused by interparticle slips, they must be proportional to increments of the rearrangement measure,  $\xi$ , and the proportionality coefficient may, in general, depend on the state of strain and stress. To express this fact, a new independent variable,  $\eta$ , may be introduced such that  $d\eta = F(\epsilon, \sigma) d\xi$ . As in the case of densification, the increment of inelastic strain per cycle diminishes as the number of cycles increases. This is true because the contact with peak interparticle shear stresses, which are the potential locations of slips, gradually become exhausted as the inelastic strain accumulates. To account for this effect, it is expedient to introduce a new independent variable,  $\zeta$ , termed intrinsic time, whose growth relative to  $\eta$  or  $\xi$  gradually diminishes. This phenomenon may be described by the relations

$$d\zeta = \frac{d\eta}{f(\eta)}; \quad d\eta = F(\epsilon, \sigma) d\eta \dots \dots \dots (12)$$

in which  $f(\eta)$ , which may be termed a strain-hardening function, is a continuously increasing positive function of  $\eta$ . Since  $d\eta$  is multiplied by an arbitrary constant,

$Z_1$ , in the stress-strain relation,  $f(0)$  can be set equal to unity. One simple choice for  $f(\eta)$  is

$$f(\eta) = \left( 1 + \frac{\beta}{r} \eta \right)^r \dots \dots \dots (13)$$

in which  $\beta$  and  $r =$  constants for a given sand. When  $\eta$  is sufficiently small,  $f(\eta) \approx 1 + \beta\eta$  for any value of  $r$ . Since  $f(\eta)$  is always positive, Eq. 12 may be integrated to yield  $\zeta = \zeta(\eta)$  and inverted to yield  $\eta = \eta(\zeta)$ . With the use of Eq. 13, Eq. 12 gives  $\zeta = (1 + \beta\eta)^{1-r} \beta r / (1 - r)$  for  $r \neq 1$  and  $(1/\beta) \ln(1 + \beta\eta)$  for  $r = 1$ . The function  $F$ , which may involve only the invariants of  $\epsilon$  and  $\sigma$ , accounts for the softening in the stress-strain relation of the material on approaching the peak (maximum) values of stress (1,3). Therefore,  $F$  may be termed a strain-softening function. However, within the range of strains commonly encountered in seismic loadings, it appears that  $F$  may be set equal to unity as an acceptable approximation.

**ENDOCHRONIC STRESS-STRAIN LAW FOR SAND**

As a consequence of material isotropy, the quasi-linear incremental stress-strain relation may be separated into one relation for the volumetric components and one relation for the deviatoric components. In the most general and plausible form of the deviatoric stress-strain relation,  $s_{ij}$  may be regarded as a functional of the past history of the stress deviator,  $e_{ij}$ . The basic assumption and most advantageous feature of the endochronic theory is that this functional may be expressed as a linear functional in terms of the intrinsic time,  $\zeta$ , because the nonlinearities are assumed to be incorporated in  $\zeta$  (as well as  $\lambda$ ). This approach is much simpler than formulating nonlinearities by means of incremental or tangential moduli, as in hypoelasticity, because these latter moduli form a fourth-rank tensor while  $\zeta$  is a scalar. The linear functional may be written as

$$s_{ij}(t) = 2 \int_0^t \bar{G}(t - t') de_{ij}(t') \dots \dots \dots (14)$$

in which the integral is a Stieltjes integral and the hereditary kernel,  $\bar{G}$ , describes the material properties. It is well known (2) that any reasonable hereditary kernel can be approximated to any desired accuracy by a series of decaying real exponentials (Dirichlet series). Since  $\zeta$  is the independent variable in which the stress-strain relation becomes quasi-linear, the sum may be written as

$$\bar{G}(t - t') = \sum_{\mu=1}^n G_{\mu} e^{-[\zeta(t) - \zeta(t')]/Z_{\mu}} \dots \dots \dots (15)$$

in which  $Z_{\mu}$  are chosen constants; one suitable choice for  $Z_{\mu}$  is  $Z_1 10^{\mu-1}$  (i.e., one  $Z_{\mu}$  is chosen for each order of magnitude of  $\zeta$ ); all orders of magnitude of  $\zeta$  that are of interest must be covered. Upon substitution from Eq. 15, Eq. 14 may be cast in the form

$$s_{ij} = \sum_{\mu} s_{ij\mu}; \quad s_{ij\mu} = 2 G_{\mu} e^{-\zeta(t)/Z_{\mu}} \int_0^t e^{\zeta(t')/Z_{\mu}} de_{ij}(t') \dots \dots \dots (16)$$

in which  $s_{ij\mu}$  are hidden stresses (internal variables). Letting  $de_{ij} = (de_{ij}/d\zeta_\mu)(d\zeta_\mu/dt)dt'$ , it may be verified that the variables,  $s_{ij\mu}$ , satisfy the following differential equations:

$$2G_\mu de_{ij} = ds_{ij\mu} + s_{ij\mu} \frac{d\zeta}{Z_\mu}; \quad s_{ij} = \sum_{\mu} s_{ij\mu} \dots \dots \dots (17)$$

which represent the deviatoric stress-strain relations. The structure of Eq. 17 may be visualized by means of a generalized Maxwell model with spring moduli  $G_\mu$ , dashpot viscosities  $G_\mu Z_\mu$  relative to intrinsic time  $\zeta$  (rather than actual time  $t$ ), and relaxation times  $Z_\mu$  on the intrinsic time scale. Because of the limited range of  $\zeta$ -values that are of practical interest, a single exponential term of Eq. 15 ought to suffice in most cases. Then,  $s_{ij\mu} \equiv s_{ij}$  and the deviatoric stress-strain relation given by Eq. 17 reduces to

$$de_{ij} = \frac{ds_{ij}}{2G} + de''_{ij}, \quad de''_{ij} = \frac{s_{ij}}{2G} \frac{d\zeta}{Z_1} \dots \dots \dots (18)$$

in which  $G$  = the shear modulus; and  $e''_{ij}$  = the inelastic deviatoric strain. Eq. 18 is similar to the constitutive relation for a single Maxwell unit.

The volumetric stress-strain relation cannot contain any inelastic volumetric strains that are dependent on volumetric stress, because purely volumetric deformations are assumed to be fully reversible. Therefore the only possible volumetric inelastic strain,  $\epsilon''$ , is the densification,  $\lambda$ ; if thermal strain were considered,  $d\epsilon'' = d\lambda + \alpha dT$ . Thus, the volumetric stress-strain relation at constant temperature is

$$d\epsilon = \frac{d\sigma}{3K} + d\lambda \dots \dots \dots (19)$$

in which  $K$  = bulk modulus of sand. The endochronic stress-strain law given by Eqs. 18 and 19 is analogous to that verified for concrete (3) and it models in a particularly simple way (without any inequalities) the irreversibility upon unloading and the hysteretic behavior of sand subjected to strain cycles. Irreversibility upon unloading, which represents the salient aspect of all inelasticity, may be illustrated by writing Eq. 18 for pure shear ( $\tau = s_{12}$ ;  $\gamma = 2\epsilon_{12}$ ) in the form  $\Delta\tau = G\Delta\gamma - G\Delta\gamma''$ , in which  $G\Delta\gamma$  is the elastic stress increment and  $G\Delta\gamma''$  is the inelastic stress increment; these increments for a prescribed value of  $\Delta\gamma$  are depicted in Figure 2. At the point where loading suddenly reverses to unloading the sign of  $G\Delta\gamma$  becomes negative, but  $G\Delta\gamma''$  does not change because  $d\zeta$  and  $d\xi = |d\gamma|/8$  do not change sign. These increments for unloading are also shown in Fig. 2, from which it is clear that that unloading branch is much steeper than the loading branch. Studies of cyclic shear test data have indicated that Eqs. 18 and 19 are capable of describing experimental results quite well within a limited range of amplitude for the shear angle (about  $10^{-4}$ - $5 \times 10^{-4}$ ), but the quantitative agreement is not as good for a broader range of amplitudes.

To extend the range of applicability, Eq. 17 has been considered with  $n = 2$  ( $\mu = 1, 2$ ),  $Z_1 < \infty$ ,  $Z_2 \rightarrow \infty$ ,  $G_1 > 0$ , and  $G_2 > 0$ ; the resulting expression, which corresponds to the stress-strain relation for a standard solid model in

intrinsic time, can still be reduced to a single first-order differential equation (without hidden stresses). Although the inelastic strain expression,  $\Delta\gamma'' = (\tau - G_2\gamma)\Delta\zeta/GZ_1$ , in this case decreases the slopes of the unloading branches at higher values of  $\gamma$  and allows somewhat improved fits in the first cycle, this model is not able to adequately describe the data over the full range of strain history because after a few cycles it yields hysteresis loops that are too narrow. One obvious improvement is to take  $Z_2 < \infty$  and add a third Maxwell unit; however, this approach is not very convenient because hidden stresses  $s_{ij\mu}$  must be introduced. As a simple expediency to resolve this problem and satisfactorily characterize all data, the shear modulus,  $G$ , may be considered a variable, rather than a constant (as implied in Eqs. 14-16). Specifically,  $G$  may be assumed to be a function of the maximum value of  $J_2(\epsilon)$  that the sand has experienced up to the current time,  $t$ ; Eqs. 14-16 require generalization in this case.

The manifested stiffening of sand with increasing hydrostatic pressure,  $-\sigma$ , is properly modeled by considering  $G$  to be a function of  $-\sigma$ . This stiffening may be explained by the fact that an increase in hydrostatic (confining) pressure prevents slip at some of the intergranular contacts due to higher normal forces, so that only elastic deformation of the grains is possible at these contacts and the microstructure becomes stiffer as a result.

Since the contact areas between grains increase with the confining stress, the solid skeleton becomes stiffer and fewer grain contacts slip as the confining stress is increased. Based on extensive experimental data for the dynamic shear modulus of sand,  $G$  has been shown to be essentially proportional to the square root of the confining stress (11). Accordingly, the shear modulus,  $G$ , may be expressed as

$$G = \sqrt{-M\sigma} \dots \dots \dots (20)$$

in which  $M$  is a material parameter that has been taken as  $7,060 \text{ MN/m}^2$  to obtain the fit of test data shown in Fig. 3; values for the other parameters are  $Z_1 = 0.5$ ,  $\beta = 1$ , and  $r = 0.7$ . These data, provided by Silver in a personal communication, pertain to cyclic shear tests on a sand at a relative density of 60% and a vertical effective stress of 4,000 psf ( $191.5 \text{ kN/m}^2$ ). It is seen that the theory correctly models both the increase in peak-strain secant (dynamic) modulus and the decrease in the area of the hysteresis loop (which is a measure of damping) as the number of cycles increases. In addition, the peak-strain secant modulus obtained from the theory can be shown to decrease as the strain amplitude increases.

From the analysis of more extensive data, however, it appears that  $M$  may be considered constant only when the range of shear strain amplitude is not too broad (definitely not exceeding one order of magnitude). For a broader range,  $M$  must be considered dependent on the strain amplitude, and because the strain amplitude may not appear in a general constitutive equation,  $M$  must be a function of the strain history. Although such a strain history dependence has been implied in the work reported by Finn, Bransby, and Pickering (8), there is no clear physical microstructural explanation for such an effect and it may be viewed as somewhat of an artifice, especially in the context of endochronic theory (this artifice implies that  $G$  depends only on one previous state, while it should depend on the entire path of strain states. Further-

more, for a given sand at different values of relative density, the material parameters will generally be functions of  $D_r$ . When more extensive data become available, it is probable that  $F(\epsilon, \sigma)$  in Eq. 12 will manifest a dependence on  $\sigma$  [perhaps similar to that found for concrete (3)], rather than the assumption of  $F(\epsilon, \sigma) = 1$ , implied herein. The reason is that hydrostatic pressure should affect both the stiffness of the microstructure and the extent of particle slippage within the microstructure. It may be possible to replace the dependence of

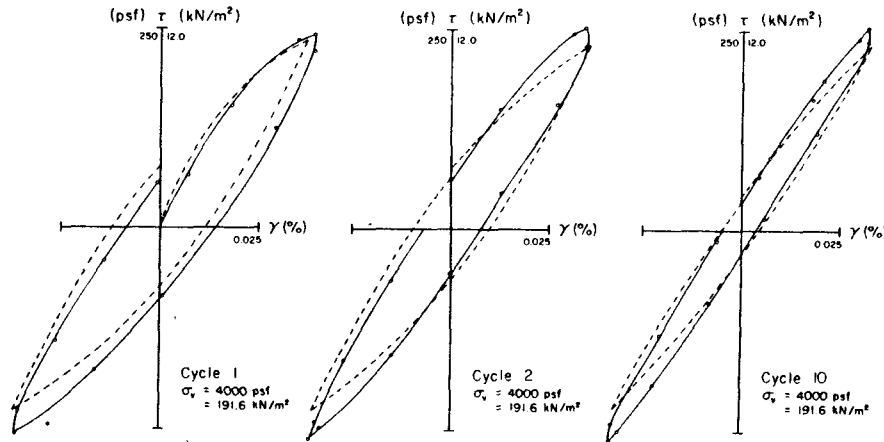


FIG. 3.—Hysteresis and Stiffening of Sand under Cyclic Shear

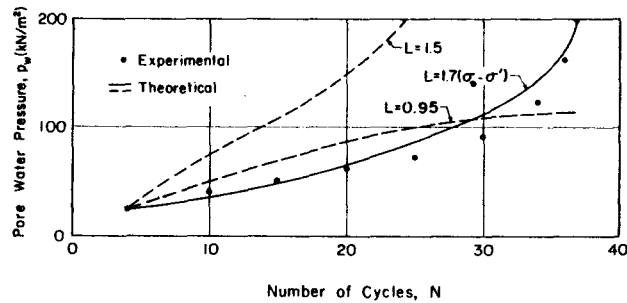


FIG. 4.—Pore-Water Pressure versus Number of Cycles

$G$  on  $\epsilon(t)$  with a dependence of  $F$  on  $p$  and  $J_2(\epsilon)$ , combined with the use of several Maxwell units in the chain. This question could be resolved by using the theory to analyze test data for small (or large) strain amplitudes followed by large (or small) strain amplitudes, or for asymmetric cycles; however, no such data are presently available. The stress-strain law given by Eqs. 18 and 19 is analogous to that verified previously for concrete (3); the only differences consist in the form of the functions,  $F$  and  $f$ , and the variation of  $G$ . The hydrostatic pressure sensitivity is a feature common with concrete and one of the major differences with regard to metals.

LIQUEFACTION OF UNDRAINED SAND

When a loose saturated sand under constant total stress is subjected to cyclic deviatoric deformations, the resulting densification causes a compression of the pore water and a buildup of the pore-water pressure, which, in turn, causes a corresponding decrease in the effective (intergranular) stress, because the total vertical normal stress in the combined solid-water system must remain constant due to a constant overburden. This decrease in the effective stress is associated with a loss of friction between the grains and a decrease in the shear modulus. In the extreme case, when the effective stress becomes zero, all friction is lost and the shear modulus is zero. Thus, the sand behaves as a liquid.

To express the interaction of sand and water, it is necessary to treat the saturated sand as a two-phase porous medium. The constitutive equation for a two-phase medium has been rigorously developed by Biot (5,6) for the case of elastic behavior, and an extension including inelastic densification has been presented by the writers (4). The deviatoric stress-strain relation for the inelastic isotropic two-phase medium has the same form as that of the solid phase alone (i.e., dry sand) but the volumetric stress-strain relation is different (see Ref. 4, Eq. 45) and may be written as

$$d\sigma = Pd\epsilon + Qd\epsilon_F - \left( P - \frac{1-n}{n} Q \right) d\epsilon'' \dots \dots \dots (21a)$$

$$d\sigma_F = Qd\epsilon + Rd\epsilon_F - \left( Q - \frac{1-n}{n} R \right) d\epsilon'' \dots \dots \dots (21b)$$

in which  $\sigma$  = the portion of the total volumetric (hydrostatic) stress,  $\sigma_t$ , carried by the solid phase;  $\sigma_F$  = the remaining portion of  $\sigma_t$  carried by the fluid phase (i.e.,  $\sigma_F = \sigma_t - \sigma = \sigma_t + np_w$ );  $n$  = the porosity;  $p_w$  = the pore-water pressure;  $\epsilon$  and  $\epsilon_F$ , defined as  $\epsilon = \text{div } \underline{u}$  and  $\epsilon_F = \text{div } \underline{u}_F$ , are the volumetric strains of the solid and fluid phases, respectively;  $\underline{u}$  and  $\underline{u}_F$  = the displacements of the solid phase and the fluid phase;  $P$ ,  $Q$ , and  $R$  = the incremental (tangent) elastic moduli of the two-phase medium; and  $\epsilon''$  = the inelastic volumetric strain in the solid phase.

In an approximate analysis of seismic liquefaction, the saturated sand may be assumed to be undrained, provided a large region of sand remains in a homogeneous state. Very often, however, this condition may not exist, since very small displacements of water can cause a large change in pressure due to the small compressibility of water. This may be important even if the duration of an earthquake is too short to allow any significant flow of water from the sand mass. Furthermore, because the overburden of any element is constant, the total volumetric stress,  $\sigma_t$ , in the two-phase medium may be considered constant, provided the ratio of lateral to vertical stress also remains constant and provided the vertical dynamic stress (whose time average is zero) is neglected. For the case where  $d\sigma_t = 0$ , the increment of pore-water pressure due to a densification increment (see Ref. 4, Eq. 48)

$$dp_w = - \frac{d\epsilon''}{C_d} \approx - \frac{d\epsilon''}{C_b} \dots \dots \dots (22)$$

in which  $d\epsilon''$  = the inelastic volumetric strain; and  $C_d$  = the densification compliance, which is given (Ref. 4, Eq. 49) by  $n^2/\{R - (Q + R)^2/(P + 2Q + R)\}$ . For typical sands in a loose or dense state,  $C_d$  is approximately equal (with an error of less than a few percent) to the bulk compressibility of drained sand  $C_b$  (Ref. 4, Eq. 50);  $C_b$  is defined as  $\partial\epsilon/\partial\sigma_1$  at constant  $p_w$ , and it is equal to  $R/(PR - Q^2)$ . It must be emphasized that  $C_d$  does not equal the bulk compressibility of undrained sand,  $C_t = (P + 2Q + R)^{-1}$ , but it is orders of magnitude larger (4). Since the total stress,  $\sigma_1$ , is constant, the change in the effective (intergranular) stress,  $\sigma'$ , at constant total stress  $\sigma_1$  is

$$d\sigma' = dp_w \dots \dots \dots (23)$$

The combination of Eqs. 22, 23, and 5 (with  $d\epsilon'' = d\lambda$ ) for an undrained sand at constant total volumetric stress yields

$$d\sigma' = -\frac{d\lambda}{C_d} = -\frac{d\kappa}{C_d c(\kappa)} \dots \dots \dots (24)$$

which implies that the solid phase of a saturated sand densifies according to the same law as the dry sand, provided the effective (intergranular) stress is the same in both cases. However, this assumption might be inaccurate for several reasons. First, the difference in friction between wet and dry grain contacts might cause deviations from the assumed conditions. These deviations are probably insignificant for small strains because the measurements of Richart, Hall, and Woods (11) indicate no appreciable difference in the shear wave velocity (and thus the shear modulus) for dry, drained, and saturated Ottawa 20-30 sand, and it is reasonable to expect that the friction between grains would behave similarly. In addition, work by Youd (14,15) shows that interparticle friction is the same for dry sand and for saturated sand under drained conditions. Nevertheless, at large shear strains the situation might be different. Another possible deviation may be due to viscous and inertia forces opposing the relative movements of grains; these movements are only microscopic, since the average macroscopic relative movement of water and sand in the undrained condition is zero. Furthermore, it should be realized that the deviator strains produce only the possibility of densification, whereas the driving force of densification is provided by the volumetric effective stress,  $\sigma'$ . This is supported by the fact that there can be no densification when  $\sigma' = 0$ . Therefore, the effective stress serves the purpose of overcoming the microscopic viscous and inertia forces. If these forces indeed affected the pore pressure buildup, its rate would have to depend on the frequency of shear cycling, and this could be used to test the hypothesis. The fact that the water present in the pores opposes densification is macroscopically manifested by the buildup of pore pressure. On the microscopic scale, the presence of water may prohibit the full realization of possible particle rearrangements that would result in densification. The actual extent of the densification realized should depend on the intergranular stress,  $\sigma'$ .

It is thus conceivable that the densification rate of a saturated sand may depend on  $\sigma'$ , even though the effect of  $\sigma$  is negligible in a dry sand. Accordingly, it might be appropriate to introduce a correction coefficient,  $L(\sigma')$ , into Eq. 24, as

$$d\sigma' = -L(\sigma') \frac{d\lambda}{C_d} \dots \dots \dots (25)$$

On the other hand, there is no reason to include the pore pressure,  $p_w$ , or the total volumetric stress,  $\sigma_1 = \sigma' - p_w$ , in this equation. As an alternative to the dependence of  $L$  on  $\sigma'$ , the effect of  $\sigma'$  on the buildup of  $p_w$  may also stem from the dependence of  $C$  (Eq. 5) on  $\sigma'$ . Although no such dependence was discerned for one type of sand (12), it may be significant for other types of sand.

Consider now a saturated sand specimen that is subjected to a sinusoidal history of shear stress,  $\tau = \tau_o \sin \omega t$ , under constant volumetric stress  $\sigma$ , all other stress components being zero. Assume further that the specimen remains always in a homogeneous state of stress and strain (for laboratory test conditions this assumption is extremely difficult, if not impossible, to satisfy due to local disturbances produced by the loading device). The tendency for liquefaction of this specimen may be assessed by integrating Eq. 27 in time and combining the relations:

$$d\gamma = \frac{1}{G(\sigma, \gamma)} \left( d\tau + \tau \frac{d\zeta}{Z_1} \right) \dots \dots \dots (26)$$

$$d\zeta = \frac{d\xi}{\left(1 + \frac{\beta}{r} \xi\right)^r}; \quad d\xi = \frac{1}{8} |d\gamma| \dots \dots \dots (27)$$

$$d\lambda = -\frac{d\kappa}{c_o(1 + \alpha\kappa)}; \quad d\kappa = \frac{1}{4} q|\gamma|^{q-1} |d\gamma| \dots \dots \dots (28)$$

which stem from Eqs. 3, 6, 8, 12, 13, 18, and 19;  $G$  is given by Eq. 20. The numerical step-by-step integration of Eqs. 25-28 was accomplished by using a computational algorithm of the same type as that described in Ref. 3; the calculation of each time step is iterated several times, first using estimated and then improved values of  $\xi$ ,  $\zeta$ , and  $\kappa$ .

This theory was applied to liquefaction data reported by Finn, Bransby, and Pickering (8) for a test conducted on a sample of loose Ottawa sand. The specimen was loaded by a controlled cyclic shear stress of amplitude 0.0245 N/mm<sup>2</sup>, and the resulting shear strains were greater than 0.001. Since hysteresis loops and the densification response for the sand in a dry state were not reported, material parameters were selected on the basis of test data for a sand with similar grain characteristics and  $\gamma_o = 0.0014$ ; only the coefficient  $L$  was considered arbitrary, and the best fit of test data was obtained with  $L = 1.7 (\sigma_o + \sigma')$ , in which  $\sigma_o = \text{constant} = 0.196 \text{ N/mm}^2$ ; and  $\sigma'$  is negative for compression ( $L$  increases as the effective stress decreases with the progress of the test). Fig. 4 shows a good agreement between experimental and theoretical results. Cases with constant  $L$  were also calculated, but a good fit could not be obtained, as shown by the dashed curves in Fig. 4.

The sharp increase in the slope of the  $p_w$ - $N$  diagram is due to the decrease in the shear modulus as the effective stress decreases. As a consequence of the reduced shear modulus, the amplitudes of the shear strain increase, and

this, in turn, causes a faster accumulation of  $\xi$ , a faster densification, and a more rapid drop in the shear modulus. This behavior is a type of feedback that is characteristic of instabilities of motion, in general.

#### CONCLUSIONS

1. Endochronic theory provides a realistic mathematical model for the densification and hysteretic behavior of sand subjected to cyclic deviatoric strains. The theory may also be applied for arbitrary strain histories, for general states of stress and strain, and for nonproportional stress or strain paths.

2. The basic extension with regard to previous applications of endochronic theory to metals is the presence of densification (negative dilatancy) and hydrostatic pressure sensitivity; these two features are shared with the previous formulation of endochronic theory for concrete.

3. The intrinsic time parameter may be associated with the rearrangement measure, which physically represents accumulated rearrangements of particle configurations.

4. The combination of the present theory with the previous formulation for inelastic densification in a two-phase medium yields a realistic model for liquefaction of an undrained saturated sand.

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#### APPENDIX II.—NOTATION

The following symbols are used in this paper:

- $C$  = densification-softening function (Eqs. 5 and 8);  
 $C_d$  = densification compliance (Eq. 25);  
 $c$  = densification hardening function (Eqs. 5 and 6);  
 $D_r$  = relative density;  
 $e_{ij}$  = deviator of strain tensor  $\epsilon$  or  $\epsilon_{ij}$ ;  
 $F$  = strain-softening function (Eq. 12);  
 $f$  = strain-hardening function (Eqs. 12 and 13);  
 $G, K$  = incremental shear and bulk modulus of sand;  
 $J_2$  = second invariant of deviator;  
 $N$  = number of cycles;  
 $s_{ij}$  = second invariant of stress tensor  $\sigma_{ij}$ ;  
 $Z_1$  = constant in Eq. 18;  
 $\gamma$  = shear angle;  
 $\epsilon_{ij}, \epsilon$  = strain tensor and its volumetric component;  
 $\zeta$  = intrinsic time (Eq. 12);  
 $\eta$  = variable in Eq. 12;  
 $\kappa$  = variable in Eq. 5;  
 $\lambda$  = inelastic densification or dilatancy (Eqs. 5 and 7);  
 $\xi$  = distortion measure (Eq. 2);  
 $\sigma_{ij}, \sigma$  = stress tensor and its volumetric component;  
 $\sigma'$  = effective volumetric stress in sand; and  
 $\tau$  = shear stress.

### Errata

"Endochronic Constitutive Law for Liquefaction of Sand" by Z. P. Bazant and R. J. Krizek, Proc. ASCE, J. Eng. Mech. Div., Vol. 102, Apr. 1976, pp. 225-238.

Replace coefficient  $1/8$  in Eq. 3 by  $1/2$ , and coefficient  $1/2$  in Eq. 4 by 2. Remove coefficient  $1/4$  from Eq. 4. Replace coefficients 2 and 8 in Eq. 8 by  $1/2$  and 4, respectively. Replace coefficient 2 on the second line after Eq. 8 by  $1/2$ . The last symbol in Eq. 12 should be  $\xi$  rather than  $\eta$ .

#### 12036 ENDOCHRONIC LAW FOR LIQUEFACTION OF SAND

KEY WORDS: Cyclic loads; Earthquake loads; Engineering mechanics; Hysteresis; Inelastic action; Liquefaction; Low pressure research; Nonlinear systems; Plasticity; Pore pressure; Sand; Saturated soils; Soil mechanics

ABSTRACT: A nonlinear constitutive law is developed, together with the previously established model for an inelastic two-phase medium, to give a realistic prediction of the pore pressure buildup and associated liquefaction of sand due to cyclic shear. The law is of the endochronic type and consists of quasi-linear first-order differential equations expressed in terms of intrinsic time, which is an independent variable whose increments depend on the strain increments. This accounts for the accumulation of particle rearrangements, which are characterized by a parameter termed the rearrangement measure. The basic extensions of this work with regard to the application of endochronic theory to metals are due to the fact that sands densify upon shearing and are sensitive to confining stress; these two features are shared with the formulation of endochronic theory for concrete. Several typical examples are given to illustrate densification, hysteresis, and liquefaction tendency of sands subjected to cyclic shear in laboratory tests.

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