Double power law for basic creep of concrete

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The dependence of creep on load duration $(t-t')$ as well as age at loading t' is described by the law $[1+\varphi_1 t'^{-m}(t-t')^n]/E_0$, in which m, n, φ_1, E_0 = material parameters which are determined from test data by optimization techniques. The law is limited to basic creep, but with different values of material parameters it can also describe drying creep up to a certain time. The previous formulations are extended by introducing the age dependence. This also enhances the reliability in long-term extrapolation of creep data. Substituting $t-t'=0.001$ day, the law also yields the correct age dependance of the conventional elastic modulus, E. If E_0, which is much larger than E, were replaced by E (as implied by previous power laws without age dependence), the age dependence of creep curves obtained by data analysis would be more scattered, the age dependence of E would have to be described by a separate formula, and more material parameters would be necessary to fit test data. The simplicity of the double power law is a major advantage for statistical evaluation of test data.

INTRODUCTION

For statistical analysis of creep data, as well as for extrapolation of the data from creep tests of limited duration and limited range of ages at loading, one needs a simple mathematical formula describing the creep curves produced by loads applied at various ages. A number of different formulas were proposed in the past (*see* Appendix). They do not differ significantly for creep durations over 1 week and up to one or a few years. Yet, for longer as well as shorter durations, striking differences are obtained. This is undoubtedly due to the limited time range of the test data used to justify some of the formulas, as well as to the differences in test conditions, especially with regard to humidity effects. Recently, Wittmann [25] has shown that a power function of the duration of creep, $\bar{t}$, originally proposed by Straub [22] and Shank [20], gives the best description of the shape of creep curves. This is especially true in the case of basic creep, i.e., when moisture exchange is negligible.

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Information on the dependence of creep upon the age t' at loading has been more scant. The purpose of this paper is to demonstrate that, in conjunction with the power law for the shape of creep curves, a power function of t' describes very closely the age effect and, in the limit, the age-dependence of the conventional elastic modulus, E, as well. Additional evidence in support of the power law for the shape of creep curves will also be presented.

PROPOSED CREEP FUNCTION AND ITS IDENTIFICATION FROM TEST DATA

Within the working stress range, the creep law of concrete may be assumed to be linear and to obey the principle of superposition [2]. Creep is then fully characterized by creep function $J(t, t')$ which represents strain ε (including the elastic strain but excluding shrinkage) in time t caused by a constant unit stress that has been acting since time t'. Time t' is measured from setting of concrete and represents its age. The following double power law (first proposed without experimental justification on page 15 of reference [2]) will be examined herein:

$$J(t, t')=\frac{1}{E_0}+\frac{\varphi_1}{E_0}t'^{-m}(t-t')^n. \qquad (1)$$

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It contains only four material parameters E_0, φ_1, m, and n.

To verify this law, the most extensive data sets available in the literature have been fitted and the optimum fits obtained are shown by the solid lines in figures 1 to 7. It is seen that the double power law indeed gives a very close description of these data. The values of material parameters for various data sets are listed in table I.

When only short-time creep data with up to one month duration and up to one month age at loading are available, all four material parameters cannot be determined from creep data. In such a case it is necessary to assume exponents m and n, whereupon determination of E_0 and φ_1 from the short-time data is normally possible. From table I it may be noticed that the differences in m and n values for various data sets are not too large. When experimental evidence is lacking, one may assume, as a mean of the values in the table I,

$$m \approx \frac{1}{3}, \qquad n \approx \frac{1}{8}. \tag{2}$$

The optimum fits have been found from the condition

$$\Phi = \sum_{r=1}^{N} \{ F_r(E_0^{-1}, \varphi_1, m, n) \}^2 = \text{Min}, \tag{3}$$

where

$$\left. \begin{aligned} F_r(E_0^{-1}, \varphi_1, m, n) &= J(t'_q+\bar{t}_p, t'_q) - \bar{J}(t'_q+\bar{t}_p, t'_q), \\ r &= (q-1) N_q + p. \end{aligned} \right\} \tag{4}$$

Here function J is given by equation (1) and $\bar{J}$ represents given experimental data on the values of the creep function at certain discrete characteristic points that are located on the creep curves and correspond to ages t'_q at loading and creep durations $t-t'=\bar{t}_p$. The values $\bar{J}$ at these points have been determined by graphical interpolation between the nearest measured values. Discrete times t'_q and $\bar{t}_p$ must be chosen in such a way that they are nearly uniformly spaced in log t'_q and log $\bar{t}_p$ scales, or else unequal weights must be attached to various terms in equation (3) for each order of magnitude of $\bar{t}_p$ and t'_q. N_p is the number of discrete times $\bar{t}_p$ and $N=N_p N_q$ where N_q is the number of discrete ages t'_q.

Equation (3) represents a nonlinear optimization problem for a sum-of-squares objective function Φ, depending on four unknown parameters E_0^{-1}, φ_1, m, and n. Perhaps the most efficient numerical algorithm for solving such problems is the modified Marquardt algorithm ([4], [13]), which has been used in the present study. A standard library program [13] is available for this algorithm, and the only computation that had to be programmed was a subroutine for the evaluation of functions F_r from specified values of E_0^{-1}, φ_1, m, and n.

AGE-SHIFT PROPERTY

It is also possible to identify the material parameters using only hand calculations. For this purpose

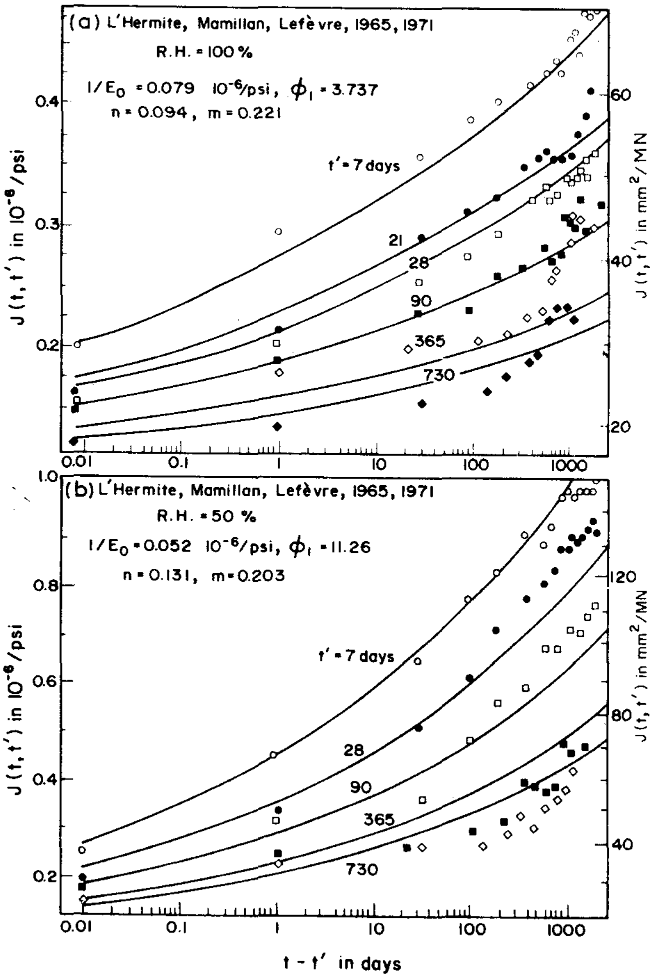

Fig. 1. L'Hermite and Mamillan's creep tests. *a*) in Water and *b*) at 50 % relative humidity. (Data constructed from reference [11], prisms 7 × 7 × 28 cm of 28-day strength 370 kgf/cm²; room temperature; concrete French type 400/800; 350 kg of cement per cubic meter of concrete; stress = 1/4 strength; water-cement-sand-gravel ratio 0.49 : 1 : 1.75 : 3.07; Seine gravel.)

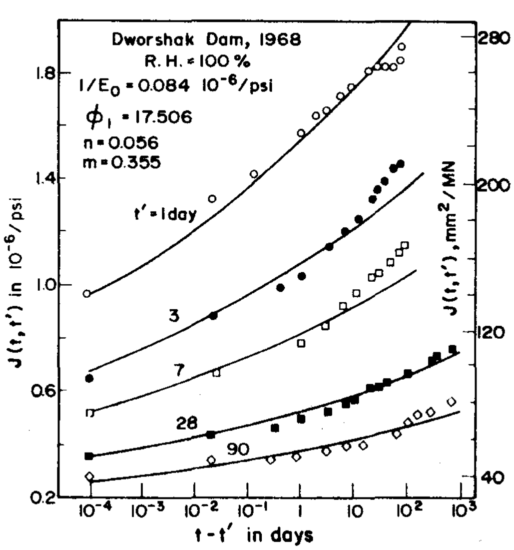

Fig. 2. — Creep tests for Dworshak Dam. (Data extracted from reference [16]; cylinders 6 × 26 inch, sealed, at 70°F; 28 day cyl. strength = 3,230 psi; stress ≤ 1/3 strength; water-cement ratio 0.58; cement type IV; max. aggregate size 1.5 inch.)

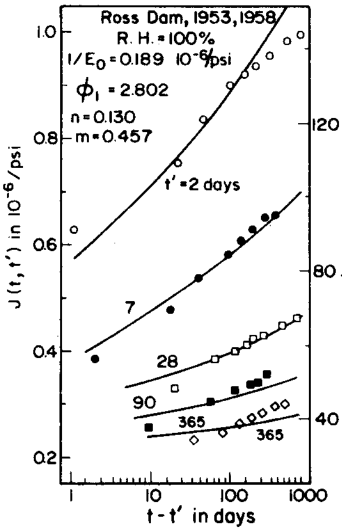

Fig. 3. Creep tests for Ross Dam. (Data extracted from references [8] and [9]; cylinders 6 × 16 inch, sealed, at 70°F; 28-day cyl. strength = 4,970 psi; stress ≤ 1/3 strength; water-cement ratio 0.56 cement type II; max. aggregate size 1.5 inch.)

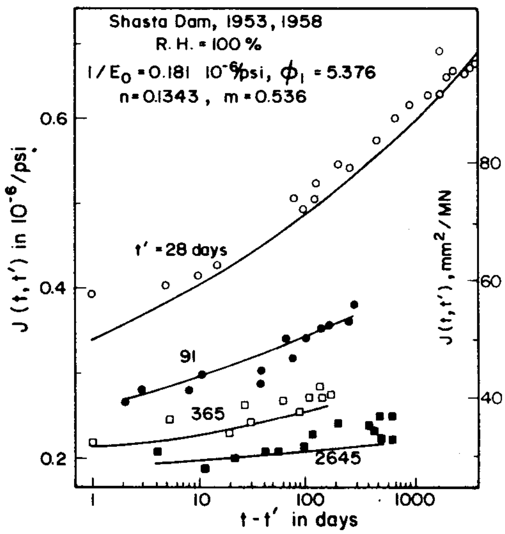

Fig. 4. — Creep tests for Shasta Dam. (Data extracted from references [8] and [9]; cylinders 6 × 26 inch, sealed at 70°F; 28-day cyl. strength = 3,230 psi; stress ≤ 1/3 strength; cement type IV; max. size of aggregate 0.75 to 1.5 inch, water-cement-sand-gravel ratio = 0.58 : 1 : 2.5 : 7.1 by weight.)

TABLE I

LIST OF DOUBLE POWER LAW PARAMETERS FOR VARIOUS CREEP DATA

Data	m	n	φ_1	$E_0^{-1} \times 10^{-6}$ psi*
1. L'Hermite *et al.* (in water)	0.221	0.094	3.74	0.0788
2. Dworshak Dam (sealed)	0.355	0.056	17.51	0.0844
3. Ross Dam (sealed)	0.457	0.130	2.80	0.1885
4. Shasta Dam (sealed)	0.536	0.134	5.38	0.1806
5. Canyon Ferry Dam (sealed)	0.295	0.119	4.02	0.1000
6. Gamble and Thomas (R.H. 94 %)	0.450	0.081	4.87	0.180
7. A. D. Ross (R.H. 93 %)	0.238	0.126	1.97	0.103
8. L'Hermite *et al.* (R.H. 50 %)	0.203	0.131	11.26	0.0521
* 1 psi = $6.895\ \mathrm{N/m^2}$.				

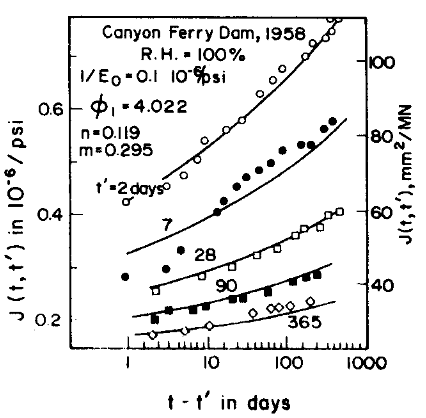

Fig. 5. Creep tests for Canyon Ferry Dam. (Data extracted from references [8] and [9]; cylinders 6 × 16 inch, sealed at 70°F; 28-day cyl. strength = 2,920 psi; stress ≤ 1/3 strength; cement type II; max. size of aggregate = 0.75 to 1.5 inch; water-cement-sand-gravel ratio 0.5 : 1 : 2.87 : 10.37 by weight.)

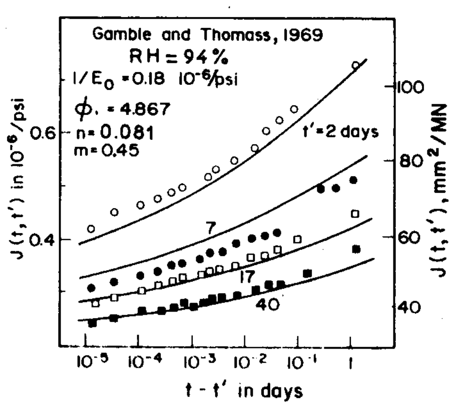

Fig. 6. Gamble and Thomass' creep tests. (Data extracted from reference [7]; cylinders 4 × 10 inch; crushed aggregate; max. size 3/16 inch; cement type I; stress-strength ratio 0.36; water-cement-sand aggregate ratio 0.7 : 1 : 2.04 : 3.06; beach sand; tested at 75°F and 94 % relative humidity.)

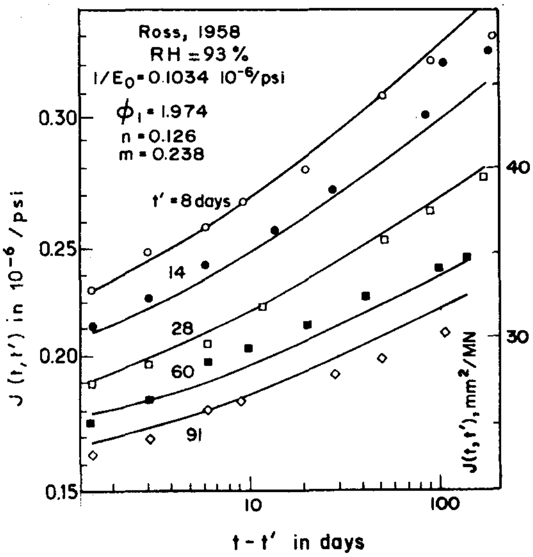

Fig. 7. — A. D. Ross' creep tests. (Data extracted from reference [19]; cylinders 4.63 × 12 inch; 28-day strength = 6,720 psi; water-cement-sand-gravel ratio 0.375 : : 1.6 : 2.8; rapid hardening Portland cement; specimens stored at 17°C and 93 % relative humidity.)

equation (1) may be rewritten in the form

$$J(t, t') = \frac{1}{E_0} + \frac{\varphi_1}{E_0} t''^n, \qquad t'' = a(t-t'), \qquad a = t'^{-m/n}. \quad (5)$$

Noting that $\log t'' = \log(t-t') + \log a$, it is evident that in log $(t-t')$- scale the creep curves for different ages t' at loading must have the same shape and be mutually shifted by distance log a in a horizontal direction (*fig.* 8 *a*). (A similar property, but with regard to temperature rather than age, is known in viscoelasticity as the time-temperature shift principle and has been exploited for concrete creep at different temperatures by Mukaddam and Bresler [15].)

In consequence of this property, a simple procedure may be used to fit test data: (*a*) One age at loading is chosen as the reference age, t'_0, and the creep curves for all other ages are shifted horizontally to conform with one smooth curve, called basic curve. (*b*) Then, noting that $-\log a = (m/n) \log t'$, the distance of shift, log a, may be plotted versus $\log t'$ and this plot may be fitted by a straight line (line $a-a$ in figure 8); the slope of this line gives ratio m/n. (*c*) Using the exact distances of shift determined from this straight line, the basic curve is corrected. (*d*) The corrected basic curve is finally fitted by the function $F(t'') = X + \varphi_1 X t''^n$ where $X = 1/E_0$. For this purpose 3 values $F_i = F(t''_i)$ $(i = 1, 2, 3)$ are selected on the corrected basic curve (at its both ends and in the middle) and the system of three equations $F_i = X + \varphi_1 X t''^n_i$ $(i = 1, 2, 3)$ must then be solved. Eliminating φ, and n from these three equations, one finds the equation

$$f(X) = X + \frac{F_3 - F_1}{t''^n_3 - t''^n_1} t''^n_2 - F_2 = 0, \qquad (6\,a)$$

with

$$n = \frac{\log(F_3 - X) - \log(F_1 - X)}{\log t''_3 - \log t''_1}. \qquad (6\,b)$$

The value of X may be solved from this equation by the regula falsi method.

The foregoing manual procedure is more tedious than the computer fitting using Marquardt algorithm. Also the fits are not as close and the m and n values found for various data show greater dispersion. Nevertheless, in addition to allowing data fitting without an automatic computer, the knowledge of the foregoing manual procedure is helpful to understand how the double power law may be exploited to extrapolate short-time creep data. With a single power law involving only creep duration there is only a relatively short curve (e. g., curve 2-2 in figure 8) available for determining exponent n, and usually this information is insufficient. In view of the age-shift property illustrated in figure 8 *a*, the double power law extends the creep curve utilizing the information available for other ages at loading, and this extension greatly enhances the accuracy in determining n. The gain in the length of creep curve due to the age-shift property is usually much larger than that due to extending a single creep test for the smallest age at loading into the age at which the last creep curve ends, and this greatly reduces the uncertainty in extrapolating creep data. (For example, in view of the age-shift property it is much more useful to carry out two creep tests for $t'_q = 15$ days and 60 da' up to the age of 90 days than it is to carry out or test for $t' = 28$ days up to the age of 120 days.)

An alternative manual fitting procedure may be based on the relation

$$\log C = \log \frac{\varphi_1}{E_0} - m \log t' + n \log(t - t'), \qquad (6\,c)$$

where C = unit creep strain $= J(t, t') - 1/E_0$. This suggests linear regression in a plot of log C versus $\log(t-t')$ and $\log t'$. However, this approach was found to yield poor fits, with higher, and erratic values for n (as well as m). This is apparently due to the fact that the value of $1/E_0$, which is needed for determining C from $J(t, t')$, must be chosen in advance, rather than optimized simultaneously. Also, the choice of $1/E_0$ appeared to be more difficult than it may seem because the $1/E_0$ – values for short load durations (1 second to 10 minutes) must be extrapolated into ultra short load duration such as 10^{-5} seconds), in order to get $1/E_0$.

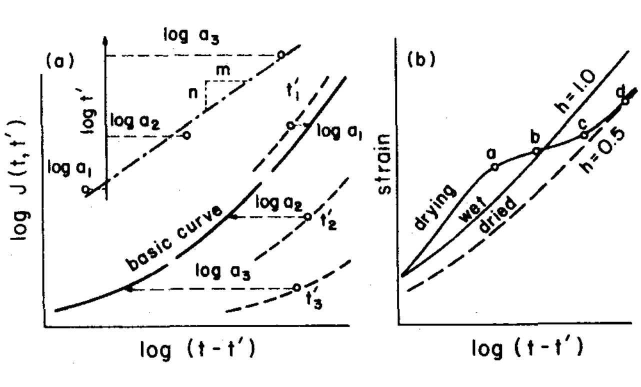

Fig. 8. — (*a*) Age-shift property; (*b*) creep curves for wet, drying, and dried concrete.

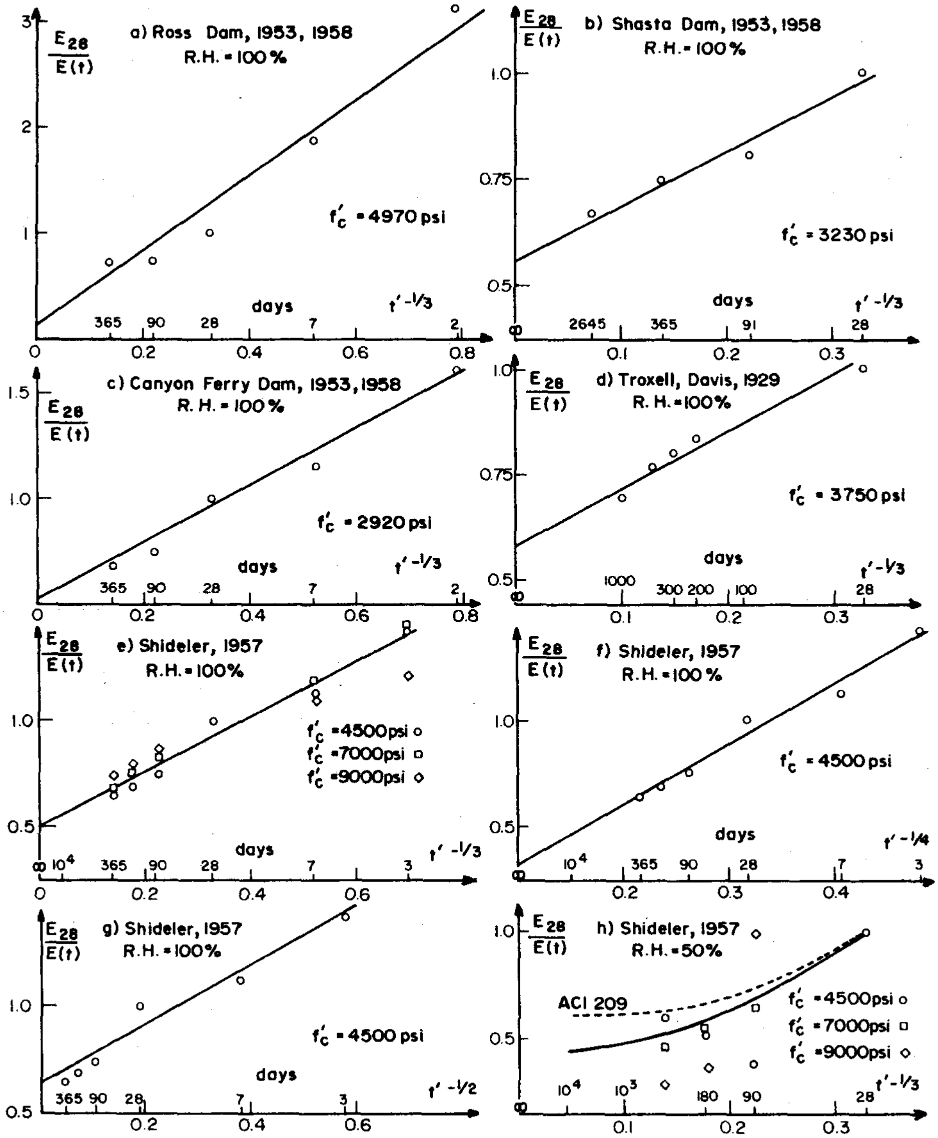

Fig. 9. — **Age dependence of elastic modulus for various concretes; (*a*)-(*c*) same test series as in figures 3-5; (*d*) Davis and Troxell's tests (reference [5]); (*e*)-(*f*) Shideler's tests (reference [21]). (Data from reference [19]: 6 × 12 inch cylinders; continuously fog-cured and tested at ages 1 to 365 days; Elgin sand and gravel (mostly quartz); water-cement-sand-aggregate ratio 0.485 : 1 : 2.8 : 4.43; 28-day cyl. strength = 4,500 psi. Data from reference [4]: 6 × 12 inch cylinders of beach sand and metamorphic sandstone, cement-aggregate ratio 1 : 3.5, max. aggregate size 1.5 inch; specimens continuously stored in damp sand; secant modulus determined at 800 psi ≦ 0.3 strength; 28-day cyl. strength 3750 psi, slump 1 inch.)**

AGE-DEPENDENCE OF ELASTIC MODULUS

Elastic modulus of concrete, as commonly understood, is not the truly instantaneous modulus, but a modulus which corresponds to loads of durations 1 to 5 minutes, or roughly 0.001 day. From figures 1, 6, and 8 it is apparent that the double power law is valid up to this time range and, accordingly, the conventional elastic modulus E is obtained from equation (1) by substituting $t-t' \approx 0.001$ day, i. e.:

$$\frac{1}{E(t')} = \frac{1+\alpha_1 t'^{-m}}{E_0}, \quad \text{with} \quad \alpha_1 = 0.001^n \varphi_1. \quad (7)$$

To further check this formula, other data ([5], [21]) on the age-dependence of elastic modulus of concrete at constant water content have been fitted by equation (7) and good agreement was found (see *fig.* 9, *d-e*).

Furthermore, the creep curves from figures 3, 4, and 5 were extrapolated to the left, considering each curve separately, without reference to the curves for other ages, and the E-values thus obtained also agreed well with equation (7) (see *fig.* 9, *a-c*). Equation (7) also agrees with the basic trend of test data of 50-year duration [24]; but for quantitative fits these data do not seem to be appropriate because of the lack of control of environmental humidity.

The exponent, m, is in equation (7) considered to be the same as m in equation (1). It has been tried to fit the test data in figure 9 (a, b, c) with various chosen values of exponent m, and for all values between 0.2 and 0.4 the fits were about equally close (see *fig.* 9, *e*, *f*, *g*). As a mean, one may assume $m \approx 1/3$ (most of the cases in figure 9). The reason for the uncertainty in determining the age exponent from the data on the elastic modulus alone is the lack of long-time data on the E-variation in saturated concrete. (There are much more data on the age-dependence of strength but their use is dubious because the ACI relation between E and the strength has not been verified for a broad range of ages.) Thus, it is also useful that, by virtue of the simultaneous consideration of creep data, equation (1) allows a much closer determination of the age exponent for the elastic modulus.

The double power law in equation (1) cannot apply for extremely short durations $t-t'$, because for $t-t' \to 0$ the value of $1/J(t, t')$ tends to the dynamic modulus E_{dyn} and according to equation (1) this equals a constant, $1/E_0$, while in reality E_{dyn} is not a constant but varies with age quite appreciably [17] though less than $E(t')$. Generalizing the double power law as follows

$$\left.\begin{aligned} J(t, t') &= \frac{1}{E_{dyn}(t')} + \frac{\varphi_1}{E_0} t'^{-m} (t-t')^n, \\ \frac{1}{E_{dyn}(t')} &= \frac{1+\alpha t'^{-m}}{E_0}, \end{aligned}\right\} \qquad (8)$$

in which the values of E_0, φ_1, m, and n are not the same as those in equation (1), the age-dependent dynamic modulus can be included in $J(t, t')$, thereby extending the validity of the expression into very short load durations $t-t'$, perhaps 10^{-4} seconds (Gamble and Thomass [7] showed that the creep curves have the shape of power curves down to the durations of 1 second). However, it was found that for $t-t'>0.001$ day it makes no sense to assume α to be non-zero because with α considered unknown the fits of the creep data in figures 1-7 (obtained by the Marquardt algorithm) were hardly any better than those for $\alpha=0$.

The long-time growth of $E(t')$ according to equation (7) is substantially higher than that according to the formula $E(t')=E_{28}[t/(4+0.85\,t)]^{1/2}$ (t in days), which is implied by the ACI Committee 209 expressions for the dependences of E on strength and of strength on age [1]. The reason is that equation (1) and (7) are intended for creep at constant water content while the ACI expression is applicable only for drying concrete specimens of standard dimensions; the hydration reaction in such specimens ceases relatively soon, whereas in concrete that is not drying the hydration continues for a long time, and this is the case to which equation (8) applies.

It is possible to extend equation (7) so that it covers both wet and drying concrete. For this purpose, t' must be replaced with the equivalent hydration period t_e [2] determined for the given humidity history. At saturation t equals t_e, while at drying conditions t_e is defined in such a way that it stops growing after concrete dries up, causing the E-curve to flatten off. An approximate formula for t_e which serves this purpose is

$$t_e = t_0 + \beta_h \hat{t} + (1-\beta_h) \frac{\hat{t}}{1+(\hat{t}/\tau_{sh})}, \qquad (9)$$

in which $\hat{t}=t-t_0$; t_0=age at the beginning of drying, and $\beta_h=[1+(10-10\,h)^4]^{-1}$ as defined on page 26 of reference 2, h=relative humidity, and τ_{sh} is the half-time of drying, i. e. the time at which the relative pore humidity in the core of the specimen changes half-way toward its final value h. (An estimation of τ_{sh} will be discussed in a separate paper dealing with shrinkage and drying creep.) Using $\tau_{sh}=30$ days, the $E(t)$–curve shown in figure 9 *h* has been obtained and its agreement with test data appears to be satisfactory. [The ACI Committee 209 formula, on the other hand, cannot be generalized so as to apply to wet concrete, and this indicates that equation (7) for $E(t')$ may be preferable.]

Another advantage of equation (7) is that by tying the $E(t')$–variation to the age-dependence of creep curves, modulus $E(t')$ for one rate of loading can be easily estimated even if the test data pertain to another rate of loading.

BASIC CREEP AND DRYING CREEP

The double power law has been found to agree well with all known data on creep at constant water content. However, in case of creep at simultaneous drying, some test data are fitted well but others are not. This is not surprising, because it is known that the simultaneous drying intensifies creep while after the drying has finished the creep rate is less the lower is the water content after drying. Therefore, the shape of the drying creep curve should be as shown in figure 8 *b*. Obviously, up to point *a* (*fig.* 8 *b*) on the drying creep curve, the data can be fitted by a power law, though with different values of material parameters than for saturation. When drying is sufficiently slow or the size of the specimen is large, point *a* in figure 8 *b* need not be reached even after 7 years of drying. This appears to be the case of the data of L'Hermite *et al.* [11] in figure 3 *b*. On the other hand, the creep curves in the test data of Troxell *et al.* [23] flatten off at the end and have apparently progressed well beyond point *a*, which is probably due to the low strength and high porosity of the concrete used (and possibly also to the particularly high carbonation rate in the initial period). Such data, of course, cannot be fitted by the power law and more complicated formulas will be necessary.

ADVANTAGES OVER OTHER CREEP LAWS

For theoretical reasons, there must be an upper age limit on the validity of equation (1), because equation (1) predicts zero creep for $t' \to \infty$, while in reality the infinitely old concrete, in which the hydration has ceased, must exhibit creep with no aging, i. e., $J(t, t')$ must become a function of only $t-t'$. To include the infinitely old concrete, equation (1) may be generalized as follows:

$$J(t, t') = \frac{1}{E_0} + \frac{\varphi_1}{E_0} (c + t'^{-m}) (t-t')^n. \qquad (10)$$

However, up to the time range of available data (30 years) the simpler equation (1) appears to be adequate.

In previous works ([20], [22], [25]) utilizing the power law (without reference to various t'), $J(t, t')$ was considered in the form $1/E+C(t, t')$, $C(t, t')=A(t-t')^n$, in which E was considered to be the conventional elastic modulus, pertaining to load duration between 1 minute and 1 hour. This value of E significantly differs from E_0 as obtained by fitting equation (1) to test data. The discrepancy between E and E_0 appears to be the chief reason for the fact that the values of the time exponent n obtained herein are distinctly less than those obtained in previous works ([20], [22], [25]; around 1/3). Another reason consists in the fact that the consideration of various t' in the present approach is equivalent to extending the time curve by several orders of magnitude. Generalization to variable t' has been also attempted with equation 1 in which E_0 is replaced by E, but the results have not been appealing because the dependence of A upon t' was more scattered and a separate expression for E as a function of t' was necessary.

In relation to the widely used logarithmic law for creep curves

$$J(t, t')=\frac{1}{E(t')}+k\log(1+t-t'), \tag{11}$$

as proposed for mass concrete by Hanson (7) (t, t' in days), it should be observed that the power law with $n>0$ always predicts higher long-time extrapolated values than the logarithmic law. [Note that the slope of the curve $(t-t')^n/n$ is $(t-t')^{n-1}$ which for $n\to 0$ becomes $1/(t-t')$, and this equals the slope of the curve $\ln(t-t')$.] For some test data the logarithmic law is about as good as equation (1) (e. g., for those in figures 3 and 5); but for other data equation (1) is distinctly better (for those in figures 1, 2 and 4).

The power-hyperbolic shape of creep curves,

$$J(t, t')=\frac{1}{E(t')}+\varphi_0 t'^{-m}\frac{(t-t')^n}{10+(t-t')^n}, \tag{12}$$

as proposed by ACI Committee 209 [1] (with $n=0.4$ to 0.8, $m=0.118$) is very close to a power function for small creep durations; but for extrapolation into long times this law yields values that are much less than those predicted by equation (1), as well as the logarithmic law. Equation (12) agrees with all data in figures 1-5 rather poorly, although it fits well drying creep curves which progressed well beyond point a in figure 8 b [23]. The main difference from equation (1) is that for long times the function in equation (12) tends to a final, asymptotic value, while the function in equation (1), as well as equation (10), is unbounded. The notion of bounded creep is comforting to designers and was perhaps one reason for the preference of various bounded creep functions in the past. The presently known data are insufficient to demonstrate boundedness of creep of concrete. Nonetheless, the trends of most data seem to indicate unbounded creep. Of course, if the creep data up to about 50-years duration are approximated well, it is practically irrelevant whether or not any final value is reached afterwards.

The argument for boundedness of concrete creep rests mainly on the test data of Troxell *et al.* [23]. Although the duration of these creep tests was the longest of all tests (30 years), these data should be used with caution because they pertain to low strength concrete which is highly porous and dries much more rapidly than contemporary concretes. (Also, according to a private communication by Professor J. Raphael, carbonation has extensively influenced the results, to a higher extent than can be expected in structures, because of smaller thickness and higher porosity of test specimens.)

Other formulas for creep curves, such as Ross-Lorman ([12], [18]) hyperbole $[m\bar{t}/(n+\bar{t})]$ or the exponential curves [6], are useful only for a rather limited time range, involving no more than two orders of magnitude of creep duration. For long-term extrapolation they greatly underestimate creep.

DIRICHLET SERIES APPROXIMATION OF DOUBLE POWER LAW

It has been shown previously ([3], [2]) that the creep analysis of large finite element systems can be tremendously simplified by expanding $J(t, t')$ into a series of exponentials, or Dirichlet series, which may be written in the form:

$$J(t, t')=\frac{1}{E(t')}+\sum_{\mu=1}^{N}\frac{1}{\hat{E}_\mu(t')}(1-e^{-[(t-t')/\tau_\mu]}) \tag{13}$$

in which τ_μ are chosen retardation times and $\hat{E}_\mu$ are coefficients. This series very closely approximates equation (1) in the time interval $0.3\,\tau_1\leqq t-t'\leqq 0.5\,\tau_N$ when one sets $\tau_\mu=10^{\mu-1}\,\tau_1$ ($\mu=1, 2, \ldots, N$) and uses the following expressions:

$$\frac{1}{E(t')}=\frac{1}{E_0}+a(n)\left(\frac{\tau_1}{0.002}\right)^n\frac{\varphi_1}{E_0}t'^{-m}. \tag{14}$$

For $\mu<N$:

$$\frac{1}{\hat{E}_\mu(t')}=b(n)\left(\frac{\tau_1}{0.002}\right)^n\frac{\varphi_1}{E_0}10^{n(\mu-1)}t'^{-m}. \tag{15}$$

For $\mu=N$:

$$\frac{1}{\hat{E}_\mu(t')}=1.2\,b(n)\left(\frac{\tau_1}{0.002}\right)^n\frac{\varphi_1}{E_0}10^{n(N-1)}t'^{-m}, \tag{16}$$

in which τ_1 and t' must be substituted in days, and $a(n)$ and $b(n)$ are coefficients given by table II. The values of $a(n)$ and $b(n)$ have been obtained by a nonlinear

TABLE II

DIRICHLET SERIES EXPANSION COEFFICIENTS [equations (14)-(16)]

n	$a(n)$	$b(n)$
0.05	0.6700	0.0819
0.10	0.4456	0.1161
0.15	0.2929	0.1229
0.20	0.1885	0.1152
0.25	0.1154	0.1007
0.30	0.0611	0.0842
0.35	0.0156	0.0681

optimization technique (Marquardt algorithm) for sum-of-squares problems. Within the time interval $0.3\tau_1 \leqq t-t' \leqq 0.5\tau_N$ the error in J^{-1} is less than $0.01(\tau_1/0.002)^n(\varphi_1/E_0)t'^{-m}$, which is so small that deviations from equation (1) can hardly be seen in a graphical plot.

CONCLUSIONS

1. The double power law [equation (1)] describes the time shape of the creep curves as well as their age dependence, and agrees quite closely with most experimental data on basic creep.

2. The double power law is limited to basic creep. However, with different values of material parameters it can be also used to describe drying creep up to a certain creep duration, which is often quite large.

3. Inclusion of age dependence in the double power law (the age-shift property) enhances the reliability in the long-term extrapolation of creep data, provided significantly different ages at loading are included among the data.

4. The double power law also yields the correct dependence of the conventional elastic modulus, E, on age.

5. Only four material parameters describe the creep curves, their age dependence, and the age dependence of conventional elastic modulus.

6. Modulus E_0 corresponds to an extrapolation of the creep curve into extremely short load durations and is much higher than the conventional modulus E. The use of E_0, which contrasts with the use of E implied in previous formulations of the power law for creep curves at a single age at loading, appears to be more advantageous. If E were used instead, the age dependence of creep curves would be more scattered; the creep law would be unable to provide at the same time the age-dependence of conventional elastic modulus E; the simple age-shift property of the double power law would be lost; and more material parameters would be necessary to fit the test data.

7. The simplicity of the double power law is a major advantage for statistical interpretation of test data, the details of which will be analyzed separately.

8. A simple relationship exists between the double power law and its approximation in terms of a series of exponentials (Dirichlet series).

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APPENDIX

List of formulas for creep curves:

The following are the best known formulas for creep curves:

$$\varepsilon(t)=\frac{1}{E}+\varphi_1\bar{t}^{\,n} \qquad (17\,a)$$

(Straub [22], Shank [20]);

$$\varepsilon(t)=\frac{1}{E}+\frac{m\bar{t}}{n+\bar{t}} \qquad (17\,b)$$

(Ross [18], Lorman [12]);

$$\varepsilon(t)=\frac{1}{E}+\varphi_u(t')\log(1+\bar{t}) \qquad (17\,c)$$

(Hanson [8]);

$$\varepsilon(t)=\frac{1}{E}+\varphi_u(t')(1-e^{-\alpha\bar{t}}) \qquad (17\,d)$$

(Dischinger [6]);

$$\varepsilon(t)=\frac{1}{E}+\varphi_u(t')\sqrt{1-e^{-\sqrt{\bar{t}/365}}} \qquad (17\,e)$$

(Mörsch [14]);

$$\varepsilon(t)=\frac{1}{E}+\varphi_1 t'^{-0.118}\frac{(\bar{t})^{0.6}}{10+(\bar{t})^{0.6}} \qquad (17\,f)$$

(ACI 209 [1]);

$$\varepsilon(t)=\frac{1}{E}+\varphi_u(t')\left\{1-\exp\left[A\bar{t}-B\log\left(1+\frac{\bar{t}}{k}\right)\right]\right\} \qquad (17\,g)$$

(L'Hermite [10]).

Remark on age-dependence of E (Added in proof)

A better confirmation of equation (7) is provided by the fit (for $m=1/3$) in figure 10 of recent long-term measurements by R. D. Browne and P. B. Bamforth (*The long term creep of the P.V. concrete for loading ages up to* 12 1/2 *years*, 3rd Intern. Conf. on Struct. Mech. in Reactor Technology, London, 1975, Paper H 1/8) (limestone aggregate, water-cement aggregate ratio 0.42 : 1 : 4.4, sealed specimen stored at 20°C, E measured on virgin specimen loaded to 14.6 N/mm², cube strength 39.5, 45.5 and 63.0 N/mm² at 7, 28 and 1,250 days).

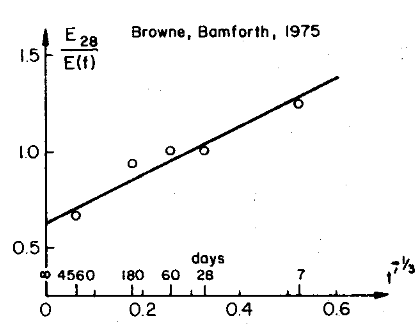

Fig. 10. — Fit of long term data on E by Browne and Bamforth.

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RÉSUMÉ

Une loi de puissance double applicable au fluage du béton en état de confinement. *La dépendance du fluage à la durée de chargement comme à l'âge du béton au temps t' de mise en charge est traduite par la loi* $\left[1+\varphi_1 t'^{-m}(t-t')^n\right]/E_0$ *où m, n,* φ_1, E_0 *sont des paramètres du matériau déterminés décrits par des techniques d'optimisation d'après les résultats d'essai. Cette loi ne s'applique qu'au béton en état de confinement (sans échanges avec l'ambiance), mais avec différentes valeurs des paramètres du matériau, elle peut rendre compte également du fluage de séchage à un moment donné. L'introduction dans les formules de la fonction d'âge recule leurs limites d'application et accroît aussi la fiabilité des extrapolations à long terme des données de fluage. Si l'on pose* $t-t'=0{,}001$ *jour, la loi fournit aussi la relation correcte à l'âge du module conventionnel d'élasticité E. Si l'on remplaçait E par* E_0 *(qui est beaucoup plus grand) — comme l'impliquent les lois de puissance antérieures sans fonction d'âge —, la fonction d'âge des courbes de fluage obtenues par l'analyse des résultats donnerait lieu à une plus grande dispersion, il faudrait alors traduire la fonction d'âge de E par une formule indépendante, et il faudrait plus de paramètres du matériau pour avoir une bonne correspondance avec les résultats d'essai. La simplicité de la loi de puissance double présente un avantage majeur pour l'évaluation statistique des résultats d'essai.*