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Comparison of Approximate Linear Methods for Concrete Creep

By Zdeněk P. Bažant,[1] M. ASCE and Leonard J. Najjar[2]

Introduction, Method of Approach, and Scope

The mathematical prediction of creep effects in concrete structures is a complicated task, even with the simplified linear forms of the creep law embodied in contemporary design recommendations (19,38). Engineers have, therefore, resorted to various approximate methods of analysis, such as the effective modulus method or the rate-of-creep method. The magnitude of the error which is due to the approximate nature of such methods has not been seriously studied, however, and intuitive judgment is widely relied upon to decide which of the various approximate methods should be used. The objective of this paper is to present the results of a detailed numerical study devoted to this question.

First it is necessary to decide on the criterion to be used for this purpose. Engineers tend to have most confidence in a direct comparison with measurements on structures. This viewpoint is undisputable when an empirical design formula for a very specific situation is under scrutiny; e.g., the camber of a composite beam of a certain cross-sectional type, certain dimensions, and reinforcement. However, generalization of such a formula to widely different cross-sectional types or dimensions, e.g., is usually unwarranted. Progress in design needs simple prediction methods of *general* applicability, and these can be achieved only through a rational theory which is based on the stress-strain law and structural mechanics. Two kinds of error in prediction of creep effects must then be discerned. One originates in the stress-strain law, and the other is due to the approximate nature of the method of analysis. Comparing the prediction directly

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[1]Prof. of Civ. Engrg., Northwestern Univ., Evanston, Ill.
[2]Grad. Research Asst., Northwestern Univ., Evanston, Ill.

with measurements on structures, provides no way to separate the two kinds of error and, therefore, no information on the accuracy of the method as such is obtained. It is thus concluded that the judgment on the relative accuracy of the various methods of creep analysis *in general* should be made primarily on the basis of comparison with the mathematically exact solutions for the creep law adopted. This approach will be followed herein.

It should not be inferred, of course, that comparisons with the measurements on structures are of little value. But for interpretation of such comparisons it is necessary to know first the degree of accuracy of the method per se. For example, if the method is known to be accurate, and yet large discrepancy with the observations on structures is found, its source must be seen in inadequacy of the stress-strain law assumed.

It must be emphasized that in many cases accurate prediction of creep effects will require a nonlinear stress-strain law. Moreover, it will be necessary to consider that the strain depends not only on the stress history but also upon the histories of water content and temperature. At present, however, such a constitutive equation is in the stage of development (16).

In the past the question has been often examined (30,41) whether the best prediction of creep effects is achieved by using the superposition principle or by using the effective modulus, rate-of-creep, or rate-of-flow methods. In the writer's opinion, however, this question has been improperly posed. All of these methods are linear and thus they also obey the principle of superposition. But it is a superposition of certain distorted unit creep curves rather than the actual measured or given ones. In a specific case, of course, one of these methods may happen to compare with measurements better than superposition of the actual creep curves. Such a case arises especially in creep recovery after sudden complete unloading, and it is the rate-of-flow method (28) which was developed to fit this case. However, complete unloading is rare in practice, and small or gradual decreases of stress, as in relaxation type problems, agree with superposition of the actual creep curves as closely as can be discerned in presence of experimental scatter (see data fits in Figs. 10 and 20 of Ref. 18). Furthermore, it is not desirable to make the creep law fit the recovery after complete unloading because the prediction of the creep of old concrete is then inevitably sacrificed, as Fig. 1(*a*) will demonstrate. It should be remembered that any deviation from the principle of superposition is a *nonlinear* effect, so that efforts to correct the deviation by means of a linear creep law that corresponds to distorted unit creep curves (such as Eqs. 21 and 22 in the sequel) are a misconception. *Stopping short of a nonlinear creep law, nothing better than superposition is possible*, in general.

Linear Creep Law of Aging Concrete and Convergent Numerical Solution

A linear creep law is known to be acceptable for stresses less than about 0.4 of the strength, provided that large reversals of strain (not stress) and repeated loads are excluded and the variation of temperature and water content is not too severe. The linearity implies validity of the principle of superposition, which yields the uniaxial relation between stress σ and strain ϵ as follows:

$$\epsilon(t) - \epsilon^0(t) = \int_0^t J(t, t')\, d\sigma(t') \quad \text{(Stieltjes integral)} \qquad (1)$$

in which t = observation time measured from casting of concrete; $J(t, t')$ = creep function (or creep compliance) = strain at time t (including elastic strain) caused by a unit constant stress acting since time t'; and ϵ^0 = stress-independent inelastic strain such as shrinkage or thermal dilatation. Eq. 1 for time-variable materials is due to Volterra (48) and was first applied to concrete by McHenry (36) and Maslov (35).

Creep is often characterized in terms of creep coefficient $\phi(t, t')$ defined as:

$$\phi(t, t') = E(t')\,J(t, t') - 1 \quad \text{or} \quad J(t, t') = \frac{1 + \phi(t, t')}{E(t')} \quad \ldots\ldots\ldots\ldots (2)$$

in which $E(t') = 1/J(t', t')$ = instantaneous elastic modulus. According

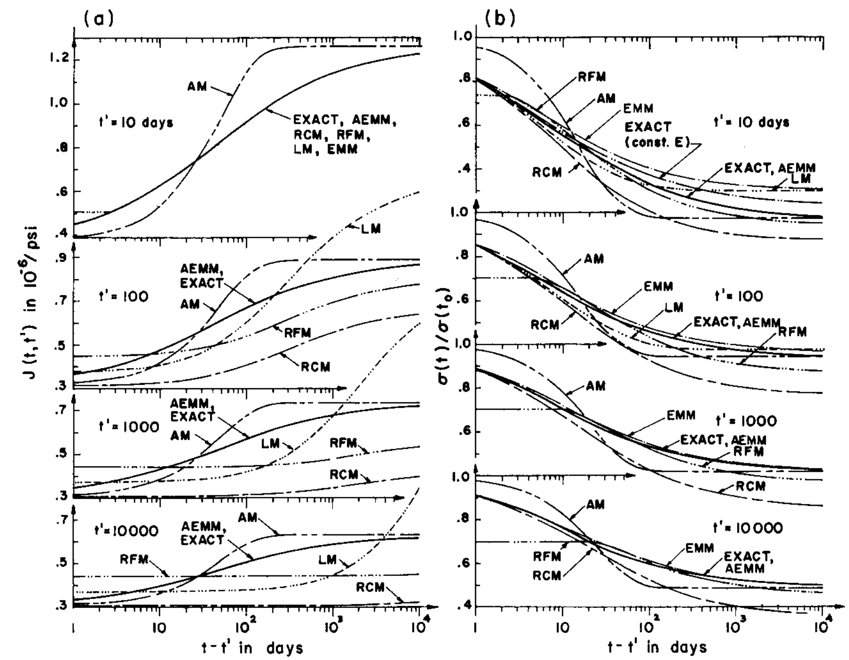

FIG. 1.—Comparisons of: (*a*) Simplified Forms of $J(t, t')$ & (*b*) Predictions of Stress Relaxation [10^{-6}/psi = 145×10^{-6}/(N/mm^2)]

to the recent recommendation of an ACI subcommittee (19,20)

$$\phi(t, t') = \phi_u(t')\,f(t - t') \quad \ldots\ldots\ldots\ldots (3)$$

$$\text{in which} \quad f(t - t') = \frac{(t - t')^{0.6}}{10 + (t - t')^{0.6}} \quad \ldots\ldots\ldots\ldots (4)$$

$$\phi_u(t') = \phi(\infty, 7)\,1.25\,t'^{-0.118} \quad \ldots\ldots\ldots\ldots (5)$$

$$\text{and} \quad E(t') = E(28)\sqrt{\frac{t}{4 + 0.85\,t'}} \quad \ldots\ldots\ldots\ldots (6)$$

t, t' being given in days. Determination of $\phi(\infty, 7)$ using a system of correction factors accounting for the effects of humidity, size, etc., is described in Ref. 19. Eq. 3, with a slightly different form of functions f and ϕ_u, has been also adopted by CEB-FIP (see Refs. 38,44). For mass concrete, Eq. 4 is better replaced by $f(t - t') = 0.113 \ln(1 + t - t')$. More complicated expressions for $J(t, t')$ which closely fit test data are given in Ref. 17.

Solution of structural analysis problems according to the creep law in Eq. 1 leads to Volterra's integral equations which can be solved only by numerical methods, of which the step-by-step methods are most efficient. For this purpose time t may be subdivided by discrete times $t_0, t_1, t_2, \ldots$ in time steps $\Delta t_r = t_r - t_{r-1}$. Time t_0 coincides with the time of loading of the structure and if the loading is steady after time t_0, times $t_0, t_1, \ldots$ are best chosen in the form of a geometric progression (11) so that in log $(t - t_0)$-scale Δt_r appears as constant. The integral in Eq. 1 may be approximated by the sum

$$\epsilon_r - \epsilon_r^0 = \sum_{q=1}^{r} \frac{1}{2}\left[J(t_r, t_q) + J(t_r, t_{q-1})\right] \Delta\sigma_q \quad \ldots\ldots\ldots\ldots (7)$$

whose error is of the order of Δt^3 (second-order method). Subscripts r, q refer to times t_r, t_q or to steps ending at these times, e.g., $\epsilon_r = \epsilon(t_r)$, and $\Delta\sigma_q = \sigma(t_q) - \sigma(t_{q-1})$. Eq. 7 is applicable even when some time step, usually Δt_1, is of zero duration ($t_1 = t_0$), i.e., the load increment is instantaneous. If r in Eq. 7 is replaced by $r - 1$ and the equation thus obtained is subtracted from Eq. 7, the following relations may be verified:

$$\Delta\epsilon_r = \frac{\Delta\sigma_r}{E''_r} + \Delta\epsilon''_r \quad \ldots\ldots\ldots\ldots (8)$$

$$\text{in which} \quad E''_r = \frac{2}{J(t_r, t_r) + J(t_r, t_{r-1})} \quad \ldots\ldots\ldots\ldots (9)$$

$$\Delta\epsilon''_r = \sum_{q=1}^{r-1} \Delta\sigma_q \Delta J_{r,q} + \Delta\epsilon_r^0 \quad \text{for} \quad r > 1; \quad \Delta\epsilon''_r = \Delta\epsilon_r^0 \quad \text{for} \quad r = 1 \quad \ldots\ldots (10)$$

$$\Delta J_{r,q} = \frac{1}{2}\left[J(t_r, t_q) + J(t_r, t_{q-1}) - J(t_{r-1}, t_q) - J(t_{r-1}, t_{q-1})\right] \quad \ldots\ldots (11)$$

The error of the method is of the order of Δt^3 (second-order method). To reduce the number of evaluations of creep function J, usually a costly part of the computations, expressions in Eqs. 9 and 11 may be replaced, without any loss in the order of error, by the expressions

$$E''_r = \frac{1}{J(t_r, t_{r-(1/2)})};$$

$$\Delta J_{r,q} = J(t_r, t_{q-(1/2)}) - J(t_{r-1}, t_{q-(1/2)}); \quad (r > 2, q > 2) \quad \ldots\ldots\ldots\ldots (12)$$

$$\text{in which} \quad t_{q-(1/2)} = t_0 + \sqrt{(t_{q-(1/2)} - t_0)(t_q - t_0)} \quad \ldots\ldots\ldots\ldots (13)$$

This is justified by the fact that under steady load stress σ varies within each time interval about linearly with log $(t - t_0)$. Eq. 12 is obviously inapplicable for the first two intervals beginning at t_0 and at $t_1 = t_0$ ($r = 1,2$, $q = 1,2$). In all numerical studies reported in the sequel, the division $t_r - t_0 = (t_{r-1} - t_0)\, 10^{0.1}$ has been used, which gave solutions that were exact to about 3 digits, as has been checked by varying the step size.

Eq. 8 has the form of an elastic stress-strain relation with elastic modulus E''_r and inelastic strain $\Delta\epsilon''$. Their values depend only on the stress increments prior to time step Δt_r and are thus known in advance, so that the solution of the increments $\Delta\epsilon_r$ and $\Delta\sigma_r$ during time step Δt_r is an elasticity problem. The only detail to be worked out is the determination of the effect of inelastic strains $\Delta\epsilon''_r$ (see Appendix I).

Eqs. 8-11 were first used in Ref. 11, in which practical convergence of the method was studied. The first general formulation of the step-by-step integration seems to have been given in Ref. 8, using the form of creep law which arises from Eq. 1 on integration by parts. But for this form of the creep law, which has thus far prevailed in numerical studies, the long-term response cannot be computed with approximations of the type of Eq. 7 because it is impossible to increase the time step beyond a certain value (related to the shortest relaxation time) without causing numerical instability. Further refinements for nonlinear creep may be found in Ref. 12 and applications to redundant structures of composite cross section and to reinforced plates in Refs. 9 and 10. The rate-of-creep analysis of large structures as formulated in Refs. 4, 5, and 15 is a special case of the method used herein.

Approximate Methods of Analysis for Linear Creep Law

In all approximate methods to be considered, the loads and enforced displacements are assumed to be steady or to vary at a rate decaying roughly as $1/(t - t_0)$; t_0 being the instant of introducing the first load or enforced deformation into the structure. Sudden load increments at various times t_0 must be analyzed separately and the results then superimposed.

Effective Modulus Method (EMM).—This is the oldest, simplest, and most widespread method (37,29,30,38). It consists of a single elastic solution using the effective (or sustained) modulus $E_{eff} = 1/J(t, t_0) = E(t_0)/[1 + \phi(t, t_0)]$. This method is known to give excellent accuracy when the aging is negligible (43), as in old concrete. In such a case, J degenerates into a function of one variable $(t - t')$; ϕ_u and E are constants; the creep curves (ϵ versus t) for loads applied at various ages t' are all identical and mutually horizontally translated; and all creep is incorrectly predicted as fully recoverable. Further, if J were, incorrectly, assumed to be bounded at $t \to \infty$, this method would give an exact solution for $t \to \infty$ (42,43). In presence of aging, the creep due to stress changes after t_0 is, obviously, overestimated.

Age-Adjusted Effective Modulus Method (AEMM).—Originated in 1967 by Trost (44), its rigorous and general formulation was first given in Ref. 14, on the basis of the following theorem (14).

Theorem.—If $\epsilon(t) - \epsilon^0(t) = \epsilon_0 + \epsilon_1\, \phi(t, t_0)$ for $t \geq t_0$, and $\sigma = 0$ for $t < t_0$, in which ϵ_0 and ϵ_1 are arbitrary constants, then

$$\Delta\sigma(t) = E''(t, t_0)\,[\Delta\epsilon(t) - \Delta\epsilon''(t)] \quad (14)$$

in which $\Delta\epsilon(t) = \epsilon(t) - \epsilon(t_0)$; $\Delta\sigma(t) = \sigma(t) - \sigma(t_0)$ (15)

$$\Delta\epsilon''(t) = \frac{\sigma(t_0)}{E(t_0)}\,\phi(t, t_0) + \epsilon^0(t) - \epsilon^0(t_0) \quad (16)$$

$$E''(t, t_0) = \frac{E(t_0)}{1 + \chi(t, t_0)\,\phi(t, t_0)} \quad (17)$$

$$\chi(t, t_0) = \frac{1}{1 - \dfrac{E_R(t, t_0)}{E(t_0)}} - \frac{1}{\phi(t, t_0)} \quad (18)$$

in which $E_R(t, t_0)$ = relaxation function = stress at time t due to constant unit strain imposed at time t_0. (The proof is given in Ref. 14.)

Because Eq. 14 has the form of Hooke's law, the computation of changes from t_0 to t due to creep is reduced to a single elastic analysis with inelastic strains, as in EMM. The most important fact is that the values of χ and E'' are independent of ϵ_0 and ϵ_1 and are thus the same for any strain history which is linear with $\phi(t, t_0)$. Such histories closely approximate the strain variation in many practical situations. Coefficient χ is called aging coefficient and E'' age-adjusted effective modulus because χ adjusts the effective modulus primarily for the aging effect; in absence of aging $\chi \approx 1$ and $E'' \approx E_{eff}$ (14). In the first studies (44,46,38,24) the term "relaxation coefficient" was used. This term was abandoned in Ref. 14 when it was found that χ has nothing to do with stress relaxation and applies for time-increasing stress regimes as well. Tables and graphs of χ for three basic forms of $J(t, t')$ have been presented in Ref. 14, along with a simple computer algorithm for their accurate computation. When different nonproportional functions $J(t, t')$ are used for different cross-sectional thicknesses, as is included in new CEB-FIP recommendations (21), this algorithm allows separate tables of χ to be easily prepared for several typical thicknesses. Always $\chi < 1$, and usually, though not always, $\chi > 0.5$. It is noteworthy that a formula for stress relaxation which is equivalent to the extreme value $\chi = 0.5$ was used by Hansen (31); the other extreme, $\chi = 1$, corresponds to EMM.

While EMM (as well as RCM, RFM, AM, LM in the following) gives an exact solution only when the stress is time-constant, this method gives an exact solution in infinitely many special cases, and especially three basic, diametrically different cases, i.e., the case of constant σ (as in the creep test), the case of constant ϵ [as in the stress relaxation test (14)], and the case $\epsilon = \epsilon_1\, \phi(t, t_0)$, typical of straining of structures by differential creep or shrinkage, and of buckling deflections. Because most other strain histories represent some kind of intermediate situation between the aforementioned cases, this method ought to be closer to the exact solution then the other approximate methods that coincide with the exact solution only in one special case (constant σ).

Rate-of-Creep Method (RCM).—This method (in German called "Dischinger's method" and in Russian "theory of aging") is due to Glanville (30) and was first applied to more complicated problems by Dischinger (25,26). It admits simple solutions by assuming the creep law in the form of a first-order differential equation:

$$\frac{d\epsilon}{d\bar{\phi}} = \frac{1}{E(t)}\frac{d\sigma}{d\bar{\phi}} + \frac{\sigma}{E(t_0)} + \frac{d\epsilon^0}{d\bar{\phi}} \qquad (19)$$

in which $\bar{\phi}(t) = \phi(t, t_0)$; t_0 being the instant the permanent load or enforced deformation is introduced into the structure. It can easily be shown (6,13,47) that Eq. 19 corresponds to an age-dependent Maxwell solid whose viscosity grows with age without bounds. Integrating Eq. 19

$$J(t, t') \approx \frac{1}{E(t')} + \frac{\bar{\phi}(t) - \bar{\phi}(t')}{E(t_0)} \qquad (20)$$

in which $\bar{\phi}$ is a function of one variable. To achieve further simplification of solutions, E is normally considered as constant. Eq. 20 is equivalent to Whitney's assumption (49) that the creep curves (ϵ versus t) for various t' are all identical but mutually vertically translated. Clearly, no delayed recovery is modeled; the creep due to stress changes after t_0 is underestimated and a negligible creep is predicted for loads applied on very old concrete. But for loads applied on a very young concrete the method gives good results, in contrast with EMM. In relaxation type problems the prediction normally lies on the other side of the exact solution than that of EMM (6,13,47); but this is assured only in the case of monotonous time-dependence of all stresses.

Rate-of-Flow Method (RFM).—Prokopovich and Ulickii (p. 37 in Ref. 40) and independently (with a much more detailed justification) England and Illston (28,32,33) proposed to represent $J(t, t')$ as a sum of delayed elastic component, which is recoverable, and flow, which is not. The flow corresponds to RCM, Eqs. 19 and 20, so that

$$J(t, t') \approx \frac{1}{E_d} + \frac{\phi_f(t) - \phi_f(t')}{E(t_0)}; \qquad \frac{1}{E_d} = \frac{1}{E(t')} + \frac{\phi_d}{E(28)} \qquad (21)$$

in which ϕ_d and ϕ_f denote creep coefficients for the delayed elastic strain and for the flow, respectively. Because the creep recovery curves at various ages are nearly identical, ϕ_d depends only on $(t - t')$ (28,40).

Thus, realizing that the effective modulus gives a very good approximation when aging is absent (14), Nielsen (39) proposed to treat the delayed elastic component in terms of the effective modulus, E_d, as defined in Eq. 21. Furthermore, for long range response, it is sufficient to take ϕ_d as constant because the creep recovery curve reaches a final value relatively soon (28). Nielsen suggested $\phi_d \approx 1/3$ and recently Rüsch and Jungwirth (21, 41 *a*) proposed $\phi_d \approx 0.4$ (for $t - t' > 90$ days). Consequently, to fit the measured or given creep curve $J(t, t_0)$, it must be assumed that:

$$\phi_f(t) = \phi(t, t_0) - \frac{\phi_d E(t_0)}{E(28)} \quad \text{or} \quad 0 \qquad (22)$$

whichever is greater, in which $\phi(t, t_0) = E(t_0)\, J(t, t_0) - 1$. To date, only very simple problems, tractable by hand superposition of creep curves, have been solved with RFM. But the assumption of constant ϕ_d makes the analytical solutions feasible, i.e., Eq. 21 is then formally equivalent to RCM and leads to the same differential equations and formulas in which effective modulus

E_d for the delayed elastic component figures instead of E. [Rüsch and Jungwirth therefore speak of "improved Dischinger's method" (21,41 *a*).] The method is a hybrid of RCM and EMM. Accordingly, it should usually give solutions that lie between RCM and EMM, similarly as the exact solution.

Levi's Method (LM).—Levi (22,34) proposed the creep law to be assumed in the form:

$$\frac{d\epsilon}{d\tilde{\phi}} + \frac{E_u}{E}\epsilon = \frac{1}{E}\frac{d\sigma}{d\tilde{\phi}} + \frac{\sigma}{E} \qquad (23)$$

which is modeled by a standard solid whose Maxwell unit has viscosity $\eta = E_m/(d\tilde{\phi}/dt)$ and elastic modulus $E_m = E - E_u$, and the parallel spring has modulus E_u; $\tilde{\phi} = \tilde{\phi}(t)$ = function of t. Integration yields:

$$J(t, t') = E^{-1} + (E_u^{-1} - E^{-1})(1 - e^{-[\tilde{\phi}(t) - \tilde{\phi}(t')]E_u/E_m}) \qquad (24)$$

Requiring that Eq. 24 fits exactly the measured or given $\phi(t, t_0) = \bar{\phi}(t)$ it is found that $\tilde{\phi}(t) = -\ln[1 - \bar{\phi}(t)\, E_u/E_m]\, E_m/E_u$, and from Eq. 24

$$J(t, t') = \frac{1}{E} + \left(\frac{1}{E_u} - \frac{1}{E}\right) \frac{\bar{\phi}(t) - \bar{\phi}(t')}{\dfrac{E}{E_u} - 1 - \bar{\phi}(t')} \qquad (25)$$

while E_u may best be determined from the condition that Eq. 25 gives the correct value of J(2,000; 1,000), as was assumed in the present study.

Arutyunian's Method (AM).—Arutyunian (2) proposed the approximation

$$J(t, t') \approx \frac{1}{E(t')} + \frac{\phi_u(t')}{E(t')}(1 - e^{-(t-t')/\tau_1}) \qquad (26)$$

where for long-time creep effects a suitable choice is τ_1 = 50 days and ϕ_u is the same as in Eq. 3. It can be shown to correspond to an age-dependent Kelvin model coupled in series with an age-dependent spring (13). It has the advantage that relaxation type problems can be reduced to first-order differential equations (with variable coefficients) for internal force rates or displacement rates (2), and a similar equation relates strain rates and stresses (13). In most problems with one unknown, analytical solutions are possible in terms of the incomplete gamma function, provided E = constant and $\phi_u/E = A + B/t'$ or $A + Be^{-t/\tau_1}$ with A, B = constants is assumed (2,13). Applications of this method have flourished (2,7,13,23,27,40,47), especially in Eastern Europe, because, in contrast with EMM and RCM, the proper ratio between the creep of young aging and old non-aging concrete can be taken into account. But the exponential shape of creep curves is far from reality [Fig. 1 (17)] and the analysis is more complicated than for other methods.

Over the years, other more complicated and rarely used approximations have also been proposed to enable analytical solutions [e.g., $J(t, t')$ on p. 223 in Ref. 1 or a relaxation function similar to Eq. 26 (3,23)].

Comparisons Between Predictions of Approximate and Exact Linear Methods

Solutions for typical practical problems have been programmed in FORTRAN

IV using the method given by Eqs. 8-13. To spare programming effort, the solutions for RCM, RFM, LM, and AM have been obtained with the help of the same programs, replacing the subroutine that supplies the values of actual $J(t, t')$ with a subroutine that supplies the values of $J(t, t')$ according to the particular approximate method. The solutions for AEMM and EMM were also obtained with the same programs, using just one long step (t_0, t), and χ was

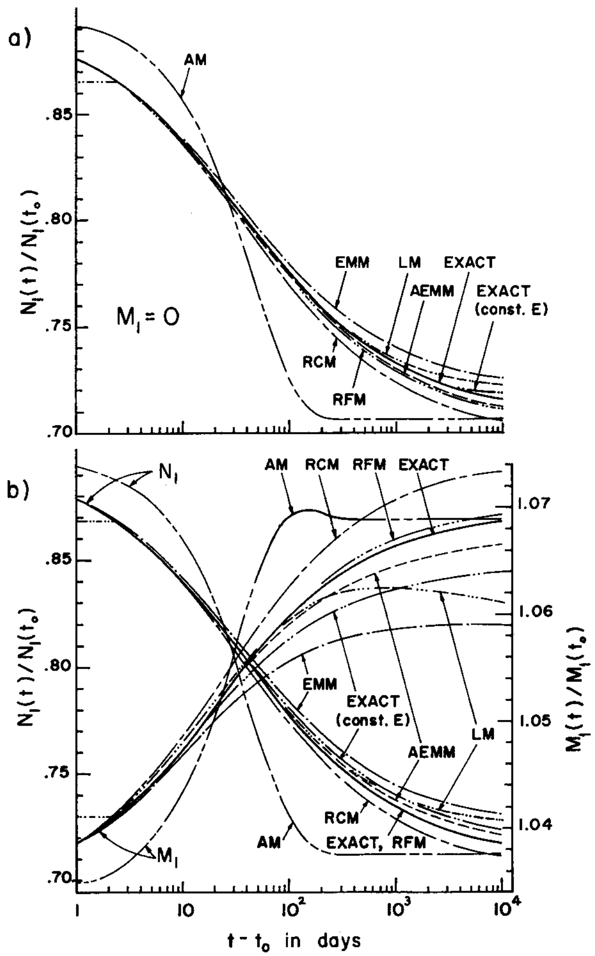

FIG. 2.—Development of Prestress Loss Due to Creep of Concrete (Including Elastic Loss)

determined by interpolation [linear in log t' as well as in log $(t - t')$] from a table of χ-values (14) computed in advance to four-digit accuracy. To assess the effect of variation of E, the exact solutions were also carried out for $E(t) = E(t_0)$ constant. The results are presented herein (Figs. 1-6) only for creep properties given by Eqs. 2-6. Creep laws for RCM, RFM, and LM were always determined as to fit exactly $\phi(t, t_0)$ for the t_0-value considered. In RFM, ϕ_d

= 0.4 was used. Predictions by AM are for ϕ_u as given by Eq. 5. All curves are plotted in logarithmic time scale. (In the literature many arguments have been supported by comparisons in the actual time scale; but this is a misleading practice because only a certain narrow time range can be seen and the results

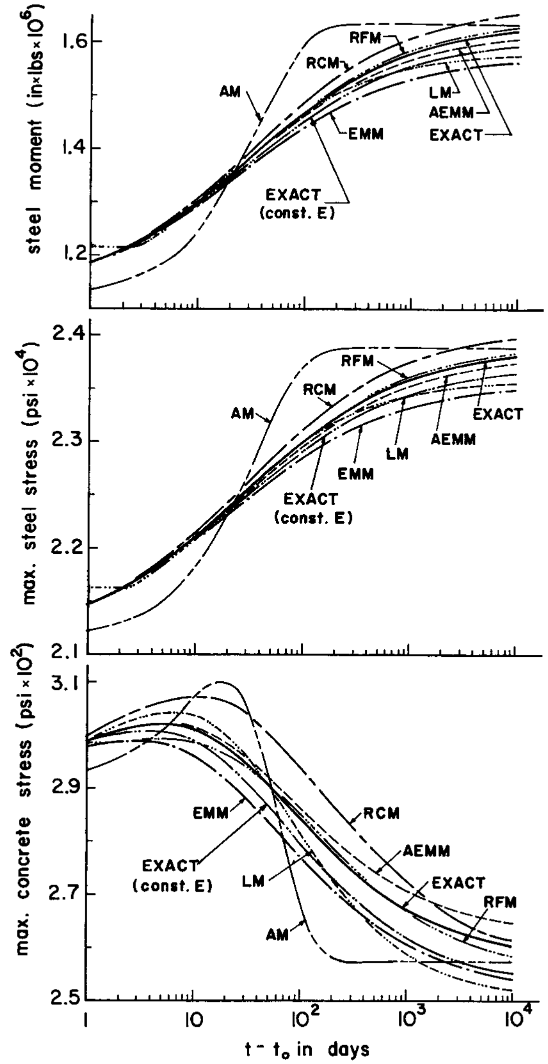

FIG. 3.—Changes of Stresses and Girder Moment Due to Creep in Steel-Concrete Composite Beam (100 psi = 0.6895 N/mm²; 10^6 in. × lb = 112.5 kN × m)

for both shorter and longer times are obscured.)

Creep Function Approximation and Stress Relaxation.—Fig. 1(a) compares the simplified forms of $J(t, t')$, as expressed by Eqs. 19 and 21, 22 and 24, and 25 and 26 with the given form: Eqs. 2-6 for $E(28) = 3 \times 10^6$ psi (21,000

Mn/m^2), $\phi(\infty, 7) = 2.5$. Note that creep deflections of most structures are roughly proportional to $J(t, t')$. The large error of RFM for high ages t' is inevitable if $J(t, t_0)$ for all t is fitted exactly.

Fig. 1(*b*) compares the predictions of stress relaxation in terms of the ratio, $\sigma(t)/\sigma(t_0)$; t_0 being the time of imposition of constant strain; $\phi(\infty, 7) = 2.5$.

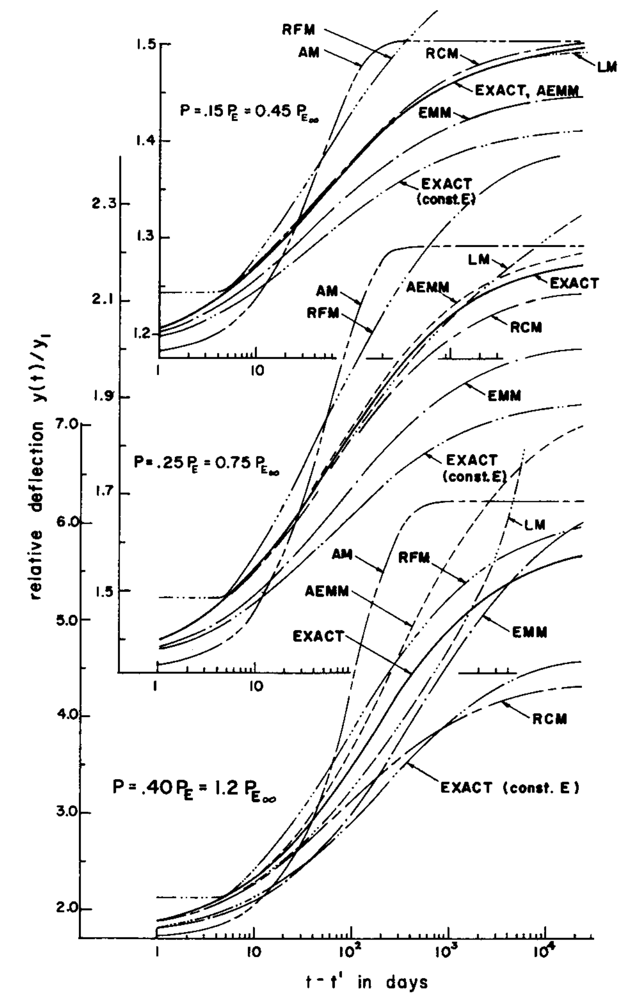

FIG. 4.—Creep Buckling Deflections of Slender Column Under Working Loads

(Creep law for both LM curves was taken for t_0 = 10 days.) This case is characteristic of stress variation caused by enforced deformations in structures, such as differential settlements or deformations imposed by jacks for the purpose of decentering (of arches) or rectifying the state of the structure (as is often done in continuous beams or box girder bridges built by cantilevering). Variation of the forces in homogeneous structures after a change in statical system (e.g.,

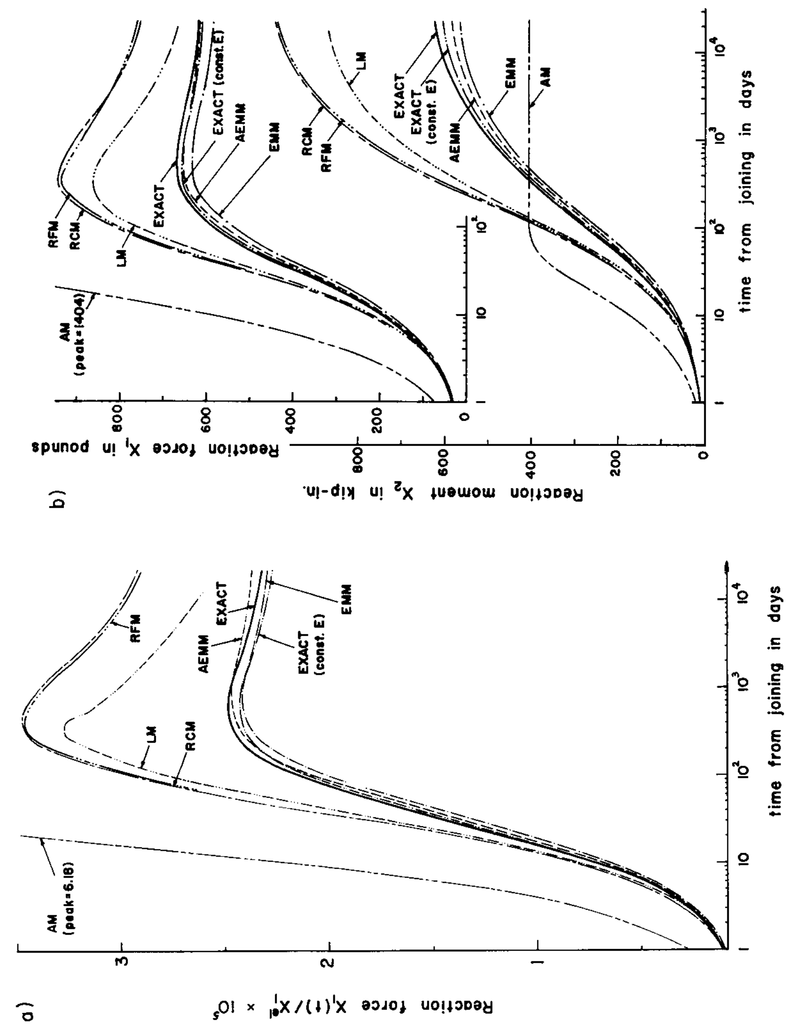

FIG. 5.—Forces in Midspan Connection of Two Cantilevers of Different Age, Due to Differential Creep (1 kip-in. = 112.5 N × m)

when a continuous beam is made by rigidly connecting simply supported beams) is also linearly related to stress relaxation function.

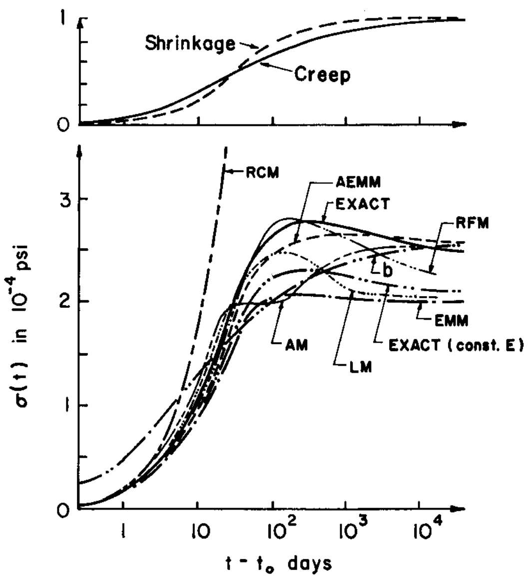

FIG. 6.—Shrinkage Stresses

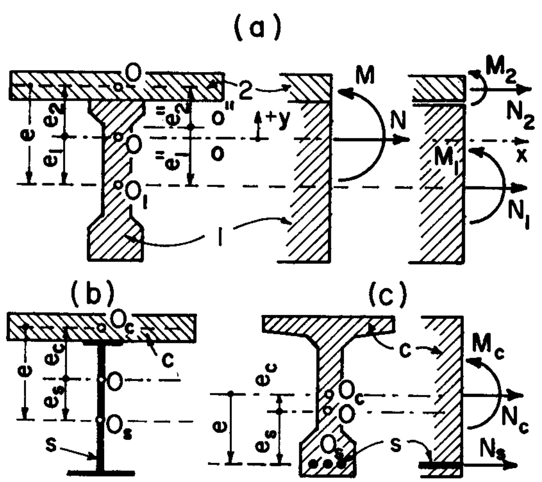

FIG. 7.—Cross Sections of Composite Beams

Prestress Loss.—Fig. 2 shows the development of prestress loss due to creep (including elastic loss) in a pretensioned statically determinate beam with a single layer of tendons (Fig. 7). In the notation defined in Appendix I and in Fig. 7, $\phi(\infty, 7) = 2.35$; $E_1 = 28.4 \times 10^6$ psi (196,000 Mn/m^2) (prestressing

steel); $E_2 = E(28) = 5.68 \times 10^6$ psi (39,100 Mn/m^2) (concrete); $I_1 \approx 0$; $A_1 = 0.186$ in.^{2} (1.195 cm^2); $e = -3.50$ in. (−88.9 mm); $I_2 = 200$ in.^{4} (8,300 cm^4); $A_2 = 22.9$ in.^{2} (147.4 cm^2); $t_0 = 7$ days; $M = N = 0$; and $N_1(t_0) = 26{,}175$ lb; (116.2 kN) = prestress before the elastic loss [while just after transfer $N_1 = 23{,}450$ lb (104 kN)]. Also included in Fig. 2 is the case of two layers of tendons (pretensioned): one on compression, one on tension side. The values of E_1, E_2, I_2, A_2, $\phi(\infty, 7)$ were the same: $I_1 = 2.437$ in.^{4} (101.2 cm^4); $A_1 = 0.248$ in.^{2} (1.596 cm^2); $e = -1.69$ in. (−42.9 mm); $M = N = 0$; $N_1(t_0) = 29{,}220$ lb (130 kN); and $M_1(t_0) = 34{,}730$ lb × in. 391 (kN × cm) [while just after transfer $M_1 = 29{,}340$ lb × in. (330 kN × cm); and $N_1 = 26{,}580$ lb (118 kN)]. Noteworthy is the fact that differences between various methods are, in the case of prestress loss, unimportant and less than the unavoidable scatter of measurements. (Therefore, e.g., good comparison in Ref. 24 of AEMM predictions with test data should not have been regarded as a verification of AEMM.)

Stress Redistributions and Deflections in Composite Beams.—Fig. 3 gives the results for a statically determinate steel-concrete composite beam (Fig. 7). Shown are the predictions of stress σ_c in the top fiber of concrete stress σ_s in the bottom fiber of steel and bending moment M_1 in the steel girder. Deflections are proportional to M_1. In the notation defined in Appendix I and in Fig. 7: $E_1 = 30 \times 10^6$ psi (21,000 Mn/m^2) (steel girder); $E(28) = E_2 = 2.52 \times 10^6$ psi (17,400 Mn/m^2) (concrete); $I_1 = 800.0$ in.^{4} (33,000 cm^4); $A_1 = 14.71$ in.^{2} (94.6 cm^2); $e = -11.0$ in. (−279 mm); $I_2 = 341.3$ in.^{4} (14,200 cm^4); $A_2 = 256.0$ in.^{2} (1,648 cm^2) and; $N = 0$, $M = 2.52 \times 10^6$ lb × in. (28,400 MN/cm^2). Value M was applied to the composite beam at $t_0 = 7$ days of age of concrete.

Creep Buckling Deflections.—Fig. 4 presents the creep buckling deflections at midspan of a slender symmetrically reinforced square concrete column (Fig. 8) with small sinusoidal initial curvature. The results are given for axial loads from 0.45 P_{E_∞} to 1.2 P_{E_∞} in which P_{E_∞} = the long-time buckling load [stability limit (7,27); and $P_{E_\infty} = P_E/[1 + \phi(\infty, t_0)]$ which equals $P_E/3.00$ for $\phi(\infty, 7) = 2.35$ and age at loading $t_0 = 28$ days, as considered in this example. The column parameters, defined in Appendix I, are: $L = 156$ in. (3,960 mm) (Fig. 9); $E_s = 30 \times 10^6$ psi (210,000 MN/m^2) (steel); $E(28) = E_2 = 6 \times 10^6$ psi (41,000 MN/m^2) (concrete); $I_s = 224.6$ in.^{4} (9,340 cm^4); and $I_c = 5{,}461$ in.^{4} (227,000 cm^4).

It is seen that for axial loads exceeding the long-time buckling load the prediction of all approximate methods becomes poor. But this case is well beyond the range of working loads.

Straining by Nonuniform Creep.—Fig. 5 shows the development of the forces in the midspan connection of two cantilevers (Fig. 9) of different age, caused by differential creep. Each cantilever was assumed to be of uniform age and the left cantilever (part 1) was assumed to be 90 days older than the right one (part 2). (In Appendix I notation $D_t^{(1)} = 0$, $D_t^{(2)} = 90$ days.) Joining was assumed to occur when the right cantilever was 90 days old ($t_0 = 180$ days). The length of each cantilever was taken as $L = 157.5$ ft (48 m) and I-values at points spaced by $L/8$ were assumed as 7,438 ft^4, 3,279 ft^4, 1,263 ft^4, 495 ft^4, and 292 ft^4 (64.2 m^4, 28.3 m^4, 10.9 m^4, 10.9 m^4, 4.27 m^4, and 2.52 m^4). (These values were taken from the design of a bridge over Vltava at Hlandná in Czechoslovakia by one of the writers.) Furthermore $E(28) = 4 \times 10^6$ psi

(28,000 MN/m^2); and $\phi(\infty, 7) = 2.5$. A simplified load history was considered, assuming each cantilever to be loaded 60 days after casting by a constant

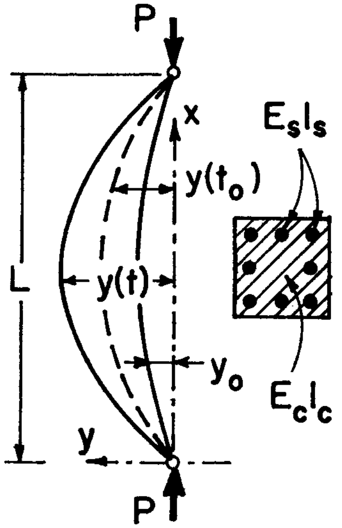

FIG. 8.—Creep Buckling of Reinforced Concrete Column

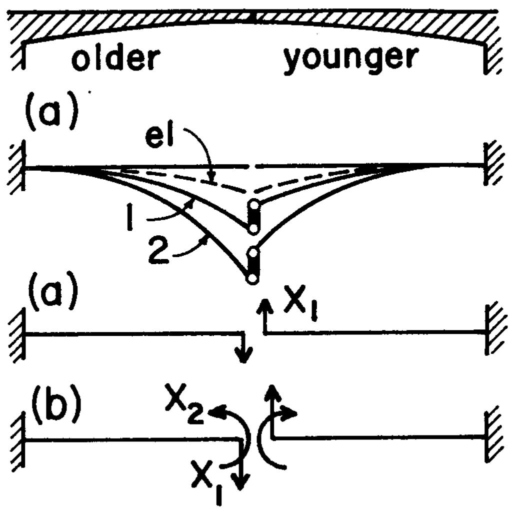

FIG. 9.—Two Cantilevers Joined at Midspan

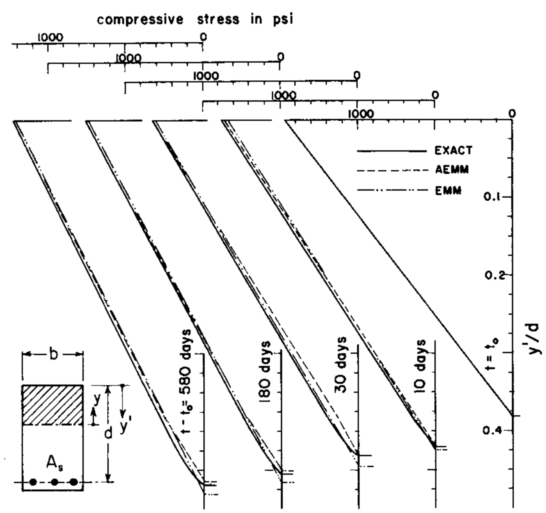

FIG. 10.—Effect of Creep on Stress Distributions in Cracked Reinforced Beam

distributed load of 10 lb/in. (1,700 N/m). Two cases were solved—cantilevers joined by a rigid hinged link [Figs. 5(a) and 9(a)] and joined monolitically [Figs. 5(b) and 9(b)]. The reactions in the connection immediately after joining were assumed as zero ($X_1 = X_2 = 0$). Integrations along the cantilevers were

carried out by Simpson's rule. It should be noted that the RCM, RFM, and LM solutions were carried out exactly for creep given by Eqs. 20-22 and 25, while in the literature all RCM applications introduced a further simplification; the so-called affinity of ϕ-curves for members of different age (see, e.g., Ref. 5), in order to facilitate analytical solutions. Value X_1^{el} = shear force at midspan due to a unit relative displacement of cantilever ends for $E = E(28)$. The exact solution for constant E is herein based on $E(28)$ rather than $E(t_0)$.

Shrinkage Stress.—Fig. 6 shows the stresses induced by shrinkage when strain is kept zero. Shrinkage is assumed to begin at $t_0 = 7$ days (beginning of drying exposure) and follow the curve $\epsilon^0(t) = 0.0008\ (t - 7)/(35 + t - 7)$ (see Ref. 19). The exact and EMM, RCM, and AEMM solutions have already been given in Ref. 14 and herein the solutions for RFM, LM, and AM are added. The solution method is given in detail in Ref. 11. Note that AEMM would give exact solution (11) if the shape of the shrinkage curve was the same as that of the creep curve; keeping $\epsilon^0(\infty)$ the same, this solution is shown as curve b. (Note that shrinkage induced forces in frames are proportional to the curves in Fig. 6.)

Cracked Reinforced Beam.—Fig. 10 reports the effect of creep upon the stress distribution in the rectangular cross section of a cracked singly reinforced concrete beam. The exact solution of this problem was obtained by Sackmann and Nickell (42). For material properties, see Ref. 42; b = 6.33 in. (161 mm); d = 10 in. (250 mm); and A_s = 1 in.^{2} (6.43 cm^2) (Fig. 10).

Conclusions

1. The proper criterion for evaluating general accuracy of the approximate methods based on a simplified linear creep function is a comparison with the exact solution according to the principle of superposition. The differences of the latter from the observed behavior of concrete can, in general, be formulated only in terms of a nonlinear creep law.

2. The age-adjusted effective modulus method (AEMM, Eqs. 14-18), along with the effective modulus method (EMM), is the simplest one because it reduces the creep problem to a single elastic analysis and, unlike the rate-of-creep (RCM), rate-of-flow (RFM), and other methods, it needs no differential equations to be solved. Also, in contrast with all other methods, it does not require any distorsion of the experimentally observed or specified unit creep curves.

3. The AEMM is overall the most accurate method. It has been checked that this conclusion applies not only for the creep function in Eqs. 2-6, but also for other creep functions of the type suitable for concrete, as mentioned after Eq. 6.

4. The RFM along with the effective modulus treatment of the delayed elastic component (Eqs. 21 and 22) is, in general, about the second best one (the exception being recovery after sudden complete unloading, which RFM is made to fit). However, one has to resort to RFM in the case when a table of the aging coefficient required by AEMM is unavailable.

5. All curves (Figs. 1-6 and Fig. 10) for RFM, as well as RCM and AM, are for variable elastic modulus E, while in practice E is usually assumed, for the sake of simplicity, as constant. This, however, introduces into these methods an additional error whose magnitude is roughly the same as the deviation

of the exact solution curves for constant E from the exact solution curves for variable E (Figs. 1-6 and Fig. 10). But for variable E the solutions of the differential equations of RFM or RCM are unacceptably complicated for practical use.

6. The differences between various methods are significant in cases of stress relaxation, shrinkage stresses, creep buckling deflections, and straining of structures by differential creep, due to nonuniformity of concrete age. In the cases of prestress loss and stress redistributions in cracked reinforced concrete cross sections, the differences are unimportant and even the effective modulus method is sufficiently accurate in comparison with the statistical scatter observed in structural behavior.

7. Contrary to a widespread opinion, the effective modulus method and the rate-of-creep method do not always give opposite bounds on the exact solution, as Fig. 4 for creep buckling demonstrates.

8. Although the Arutyunian's method is mathematically most complicated, it is far from best. But its final values are much better than those of EMM or RCM.

9. Prediction of the complete time dependence is important since maximum stress may occur at a finite time (Figs. 3-6).

10. This study also shows that, in problems with a small number of unknowns, exact solutions for a linear creep law can be easily obtained.

On the basis of the present results, the age-adjusted effective modulus method has been proposed by the writers to be adopted by American Concrete Institute Committee 209 for its recommendations and the CEB (European Concrete Committee) Working Group for its manual for creep and shrinkage analysis currently under preparation.

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Appendix I.—Details of Computation of Creep Effects

Composite or Prestressed Beams (Uncracked).—Consider a general composite beam made of two parts, 1 and 2, each of which exhibits different linear creep properties. As a special case, one part may consist of steel, i.e., it has zero creep. Quantities referring to individual parts will be referred to by subscript i ($i = 1, 2$). Denote N_1, N_2, M_1, M_2 = normal forces and bending moments in part 1 or 2 about its centroidal axis; A_1, A_2, I_1, I_2 = areas and moments of inertia of cross section of part 1 or 2 about its centroidal axis; e_1, e_2 = eccentricities of centroids 0_1, 0_2 of Parts 1 and 2 from point 0 on the chosen beam axis-x, positive when the vectors $\mathbf{OO}_1$, $\mathbf{OO}_2$ point upwards, otherwise negative; $e = e_1 - e_2$ = distance between centroids of parts 1 and 2, positive when $\mathbf{O}_2\mathbf{O}_1$ is oriented upwards, otherwise negative; N, M = normal force and bending moment about $\mathbf{O}$ in the whole cross section; k = curvature of beam; ϵ_1, ϵ_2, ϵ_0 = normal strains in centroids $\mathbf{O}_1$, $\mathbf{O}_2$; and point $\mathbf{O}$ on axis x (see Fig. 7).

The inelastic strain increments, $\Delta\epsilon''_a$, $\Delta\epsilon''_b$, are distributed linearly within each part and are equivalent to inelastic increments of curvature and strains in centroids, which are expressed, in analogy with Eqs. 10

$$\Delta k''_{i_r} = \sum_{q=1}^{r-1} \frac{\Delta M_{i_q} \Delta J_{i_{r,q}}}{I_i}; \qquad \Delta\epsilon''_{i_r} = \sum_{q=1}^{r-1} \frac{\Delta N_{i_q} \Delta J_{i_{r,q}}}{A_i} \quad (i = 1,2) \quad \ldots\ldots\ldots (27)$$

The equivalent inelastic bending moments and normal forces are

$$\Delta M''_i = E''_{i_r} I_i \Delta k''_i; \qquad \Delta N''_i = E''_{i_r} A_i \Delta\epsilon''_i \quad (i = 1, 2) \quad \ldots\ldots\ldots (28)$$

in which E''_{i_r} = the values given by Eq. 9. The resultants over the whole cross section are

$$\Delta M'' = \Delta M''_1 + \Delta M''_2 - \Delta N''_1 e_1 - \Delta N''_2 e_2; \qquad \Delta N'' = \Delta N''_1 + \Delta N''_2 \quad \ldots\ldots (29)$$

Then, if the beam is statically determinate (otherwise see Ref. 9), the changes in curvature and axial strains during time interval Δt_r are

$$\left.\begin{aligned} \Delta k_r &= \frac{\Delta M_r + \Delta M'' - \sum_i (e''_i - e_i)(\Delta N + \Delta N'')}{R''_M} \\ \Delta\epsilon_{i_r} &= \frac{\Delta N_r + \Delta N''}{R''_N} - e''_i \Delta k \qquad (i = 1, 2) \end{aligned}\right\} \quad \ldots\ldots\ldots (30)$$

in which

$$R''_N = \sum_i E''_{i_r} A_i;\ e''_1 = \frac{e E''_2 A_2}{R''_N};\ e''_2 = e''_1 - e; \quad R''_M = \sum_i E''_{i_r}(I_i + A_i e''^2_i) \quad (31)$$

and ΔM_r, ΔN_r are given changes of M and N from time t_{r-1} to time t_r (zero in the cases studied herein). Finally, the changes of M_i, N_i during time interval Δt_r are

$$\Delta M_{i_r} = E''_{i_r} I_i \Delta k_r - \Delta M''_i; \qquad \Delta N_{i_r} = E''_{i_r} A_i \Delta\epsilon_{i_r} - \Delta N''_i \quad (i = 1, 2) \quad \ldots (32)$$

The foregoing procedure (Eqs. 27-32) is based on a general theorem given in Refs. 12, 16 or 41 where a more detailed description may be found. It was first applied in Ref. 5, in conjunction with the rate-of-creep method.

In the examples of prestress loss (Fig. 2) and the composite steel girder (Fig. 3) part 1 represented steel and was assumed with zero creep.

Creep Buckling Deflections of Columns.—Small deflections of a hinged slender symmetrically reinforced concrete column of length L has been studied (Fig. 8). The column is loaded by a constant axial force P (positive for compression) applied at time t_0. Assume that before loading, $t < t_0$, the column has a slightly curved shape, $y_0 = a \sin(\pi x/L)$, x being the length coordinate. Denote $E_s I_s$, $E_c I_s$ = bending rigidities of steel and concrete cross sections; k = curvature of column; M_c = bending moment carried by net concrete cross section; and y = deflection ordinate (measured from the chord). In analogy with Eqs. 8 and 10 for $\Delta\epsilon''$:

$$\Delta k_r = \frac{\Delta M_{c_r}}{E''_{c_r} I_c} + \Delta k''_c, \qquad \Delta k''_c = \sum_{q=1}^{r-1} \Delta M_{c_q} \frac{\Delta J_{r,q}}{I_c} \quad \ldots \quad (33)$$

in which E''_{c_r} is given by Eq. 9. Equilibrium of bending moment increments at constant P requires that:

$$\Delta M_{c_r} + E_s I_s \Delta k_r = -P \Delta y_r \qquad (\Delta P = 0) \quad \ldots \quad (34)$$

Eqs. 33 and 34 may be satisifed identically if both Δy_r and Δk have sinusoidal distributions, i.e., $y \sim \sin(\pi x/L)$. Then $\Delta k_r = d^2(\Delta y_r)/dx^2 = -\Delta y_r \pi^2/L^2$, and substitution for Δk_r in Eqs. 33 and 34 furnishes:

$$\Delta y_r = \frac{E''_{c_r} I_c \Delta k''_{c_r}}{P - (E''_{c_r} I_c + E_s I_s)\dfrac{\pi^2}{L^2}}, \qquad \Delta M_{c_r} = \left(E_s I_s \frac{\pi^2}{L^2} - P \right) \Delta y_r \quad \ldots \quad (35)$$

Computation in each time step proceeds through Eqs. 34 and 35 and values Δy_r, ΔM_{c_r} are evaluated only for the midspan. Note that values ΔM_{c_r} from all previous time steps must be stored. The starting value due to initial elastic deflection at time t_0 ($r = 0$) is

$$\Delta M_{c_0} = \frac{P y_0}{1 - \dfrac{P}{P_{E_0}}} \quad \text{in which} \quad P_{E_0} = [E_c(t_0) I_c + E_s I_s] \frac{\pi^2}{L^2} \quad \ldots \quad (36)$$

The AEMM seems not to have yet been applied to creep buckling. Its formula is obtained as a single time step described in Eq. 35 and reads:

$$\frac{y(t)}{y(t_0)} = 1 + \frac{P}{P''_E - P} \frac{\phi(t, t_0)}{1 + \chi(t, t_0)\,\phi(t, t_0)} (1 - \rho_s) \quad \ldots \quad (37)$$

$$\left.\begin{aligned} &\text{in which} \quad P''_E = [E''_c(t, t_0) I_c + E_s I_s] \frac{\pi^2}{L^2} \\ &\rho_s = \frac{E_s I_s}{E_c(t) I + E_s I_s}, \qquad y(t_0) = \frac{y_0}{1 - \dfrac{P}{P_{E_0}}} \end{aligned}\right\} \quad \ldots \quad (38)$$

Nonhomogeneous Structures of Nonuniform Age.—Time t will now be viewed as a common reference age and $t^{(p)}$ will denote the age of part p of the structure ($p = 1,2,\ldots$). Let $t^{(p)} = t - D_t^{(p)}$, in which $D_t^{(p)}$ = given age difference with respect to the chosen reference age.

Consider a statically indeterminate structure (e.g., Fig. 9) consisting of beams whose cross sections are, for the sake of simplicity, taken as homogeneous, with creep properties given by a certain creep function, $J(t, t')$. (For nonhomogeneous cross sections see Ref. 9.) Using a step-by-step analysis, the inelastic curvature increments during time step Δt_r may be expressed, in analogy with Eq. 10:

$$\Delta k''_r = \sum_{q=1}^{r-1} \frac{\Delta M_q \Delta J^{(p)}_{r,q}}{I} \quad \ldots \quad (39)$$

in which superscript p refers to part p and I = centroidal inertia moments of cross sections. Adopting the force method of analysis, the corresponding deformations of the primary system in the sense of the chosen redundants, X_i ($i = 1,2,\ldots,N$), and the flexibility coefficients for time step Δt_r, are:

$$\Delta a_{i_r} = \int_{(x)} \bar{M}_i \, \Delta k''_r \, dx; \qquad f''_{ij_r} = \int_{(x)} \frac{\bar{M}_i \bar{M}_j}{E''_r I} dx \quad \ldots \quad (40)$$

in which x = length coordinate of beams; $\bar{M}_i$ = bending moments due to state $X_i = 0$, $X_j \equiv 0$ for all $j \neq i$; and E''_r are the moduli given by Eq. 9 for the proper age of concrete in the cross section. Further, denote ΔX_{j_q} = changes of redundants over time steps Δt_q, which must be stored for all steps; and ΔM^L_q = changes of moments due to changes in given applied loads from time t_{q-1} to t_q. Substituting $\Delta M_q = \Delta M^L_q + \Sigma_j \bar{M}_j \Delta X_{j_q}$ into Eq. 39 and Eq. 39 into Eq. 40, it can be found that

$$\Delta a_{i_r} = \sum_p \sum_{q=1}^{r-1} \left(\Delta a^{L(p)}_{i_q} + \sum_j f^{(p)}_{ij_q} \Delta X_{j_q} \right) E^{(p)}_q \Delta J^{(p)}_{r,q} \quad \ldots \quad (41)$$

$$\text{in which} \quad \Delta a^{L(p)}_{i_q} = \int_{(x)} \frac{\bar{M}_i \Delta M^L_q}{E_q I} dx \quad \ldots \quad (42)$$

Similarly, using Eq. 9

$$f''_{ij_r} = \sum_p f^{(p)}_{ij_r} E^{(p)}_r J^{(p)}(t_r, t_{r-(1/2)}) \quad \ldots \quad (43)$$

in which $f^{(p)}_{ij_r}$ and $\Delta a^{L(p)}_{i_q}$ = the flexibilities and deformations due solely to part p. They are given by Eqs. 40 with modulus $E(t^{(p)}_r)$, the integrals extending only over part p. The changes of redundants during step Δt_r are solved from the equations

$$\sum_j f''_{ij_r} \Delta X_{j_r} + \Delta a_{i_r} = 0 \quad \ldots \quad (44)$$

In each time step the computation proceeds over Eqs. 42, 41, 43 and 44. The method is obviously also applicable even when $J(t, t')$ is different for different cross sections.

Cracked Reinforced Concrete Beam (Statically Determinate).—The neutral axis is known to move, under permanent load, downwards, and the stress distribution between the original and the current position of neutral axis ($y = 0$) becomes nonlinear. To describe this effect at least approximately, the distribution of stress below and above the current neutral axis has been assumed according to straight lines of different slopes. The stress distribtuion is then characterized by stresses σ_0 at $y = 0$ and σ_1 at some chosen point y_1, and by the distance,

c_n, from point y = 0 to the final position of the neutral axis. Applying Eq. 14 of AEMM, the strain increments from time t_0 to time t at points y = 0 and y = y_1 are

$$\Delta\epsilon_0 = \frac{\Delta\sigma_0}{E''}; \qquad \Delta\epsilon_1 = \frac{\Delta\sigma_1}{E''} + \frac{\sigma_1(t_0)}{E(t_0)}\phi(t, t_0) \quad \ldots\ldots\ldots\ldots\ldots \quad (45)$$

Further it is necessary to add one elastic relation for the stress increment in steel, two equations expressing that the strains in steel and in points y = c_n, y = 0, y = y_1 must be linearly distributed, and finally two equations of equilibrium within the cross section. Thus seven algebraic equations for seven unknowns $\Delta\epsilon_0$, $\Delta\sigma_0$, $\Delta\epsilon_1$, $\Delta\sigma_1$, $\Delta\epsilon_s$, $\Delta\sigma_s$, c_n are obtained, which can be reduced to one cubic equation for c_n. The solution must be carried out in an iterative fashion. For the choice of point y = y_1 various locations have been tried. It was found that for a rectangular cross section the best results are obtained when this point coincides with y = 0.8c (the case shown in Fig. 10), rather than the top fiber $y = c$.

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Appendix III.—Notation

The following symbols are used in this paper:

$A_1, A_2, I_1, I_2, e_1, e_2, e,$
$A_c, A_s, I_c, I_s, e_c, e_s$ = areas, inertia moments, and eccentricities of parts of composite cross section (Fig. 7, Appendix I);
a_i, f_{ij} = deformations due to loading and flexibility coefficients (Eqs. 40, 45);
$D_t^{(p)}$ = reference age minus age of part p (at same instant);
E'' or E''_r = pseudo-instantaneous modulus (Eq. 11*a*) or age-adjusted effective modulus (Eq. 14);
E_c, E_s = moduli of concrete and of steel in composite structure;
E_u, E_m = moduli in Eqs. 23, 24;
$E(t) = 1/J(t, t)$ = elastic modulus of concrete in time t;
$f(t - t')$ = function of $t - t'$ in Eq. 3;
$J(t, t')$ = creep function (or creep compliance) defined after Eq. 1;
k, $\Delta k''$ = curvature of beam and its inelastic increment;
$M, N, M_1, N_1, M_2, N_2,$
M_c, N_c, M_s, N_s = bending moment and normal force in cross section and in its Parts 1 and 2 or concrete and steel parts (Fig. 7, Appendix I);
$\bar{M}_i$ = M in primary system due to $X_i = 1$;
P_E, P_{E_∞} = instantaneous and long-time buckling load (stability limit) of column;
t, t' = time from casting of concrete, in days;
$t = t'$ = time elapsed from loading;
t_0 = instant of introducing first load or enforced deformation into structure;
t_r, t_q, $t_{q-1/2}$ = chosen discrete times for step-by-step analysis ($r,q = 1,2,3,\ldots$), and time given by Eq. 9;
X_i = statically indeterminate quantities;
y = column deflection (at midspan in Fig. 4) or depth coordinate in cross sections;
Δ = increment sign for interval t_{r-1}, t_r; e.g., $\Delta\epsilon_r = \epsilon_r - \epsilon_{r-1}$;
$\Delta J_{r,s}$ = expression defined by Eq. 12;
$\Delta\epsilon''$ = pseudo-inelastic strain increment in Eq. 11*b* or Eq. 14;
ϵ, ϵ^0 = total strain and stress-independent inelastic strain (shrinkage, e.g.);
$\phi(t, t')$ = creep coefficient (Eq. 2); and
ϕ_f, ϕ_u, $\bar{\phi}$, $\tilde{\phi}$ = coefficients defined by Eqs. 21, 3, 19, 23.

Subscripts

c, s = for concrete and steel;
i = 1,2 for parts 1 and 2 or composite section (Fig. 7); and
r, q = for discrete times r, q, $\epsilon_r = \epsilon(t_r)$, $J_{r,q} = J(t_r, t_q)$, etc.

Superscript

(p) = for part p of structure (Appendix I).

10006 APPROXIMATE LINEAR METHOD FOR CONCRETE CREEP

KEY WORDS: **Aging tests (materials);** Approximation method; Beams (supports); Bridges (structures); Composite beams; **Concrete; Concrete (prestressed);** Concrete (reinforced); **Creep;** Creep rate; Deflection; Design practices; Numerical analysis; Predictions; Shrinkage; Stresses; **Structural engineering**

ABSTRACT: The approximate methods for prediction of structural effects of creep, such as the effective modulus (EMM), age-adjusted effective modulus (AEMM), rate-of-creep (RCM), rate-of-flow (RFM), and Levi's and Arutyunian's Methods are all linear and satisfy the principle of superposition. Predictions of the approximate methods are compared with the "exact" numerical solutions based on the principle of superposition and given (undistorted) realistic unit creep curves. It is found that AEMM is in general superior to all other methods and along with EMM is also the simplest one RFM (with effective modulus treatment of delayed elastic strain) appears as second best and should be resorted to when the table of aging coefficient required by AEMM is unavailable.

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