

international journal of
fracture

Volume 137, Nos. 1–4, JANUARY – FEBRUARY 2006

PUBLISHED TWICE MONTHLY

TABLE OF CONTENTS

Special Issue

IN HONOR OF PROFESSOR ZDENĚK P. BAŽANT

Guest Editor

GEORGE J. DVORAK

K. RAVI-CHANDAR, Editor-in-Chief
The University of Texas at Austin

Published since 1965

 Springer

international journal of
fracture

Vol. 137, Nos. 1-4, 2006

TABLE OF CONTENTS

Preface		1-7
A probabilistic method to predict fatigue crack initiation	S.S. KULKARNI, L. SUN, B. MORAN, S. KRISHNASWAMY and J.D. ACHENBACH	9-17
Application of particle methods to static fracture of reinforced concrete structures	T. RABCZUK and T. BELYTSCHKO	19-49
A CDM approach of ductile damage with plastic compressibility	J.L. CHABOCHE, M. BOUDIFA and K. SAANOUNI	51-75
Stress intensity controlled kinetic crack growth and stress history dependent life prediction with statistical variability	R. CHRISTENSEN and Y. MIYANO	77-87
The influence of a surface coating on the high-rate fragmentation of a ductile material	P.R. GUDURU, M.S. BHARATHI and L.B. FREUND	89-108
Size sensitivity for the reliability index in stochastic finite element analysis of damage	M.A. GUTIÉRREZ	109-120
Solution of multiple edge cracks in an elastic half plane	X. JIN and L.M. KEER	121-137
A computational damage micromodel of laminated composites	P. LADEVÈZE, G. LUBINEAU and D. VIOLEAU	139-150
Health assessment of concrete dams by overall inverse analyses and neural networks	R. FEDELE, G. MAIER and B. MILLER	151-172
Effects of microstructure variability on intrinsic fatigue resistance of nickel-base superalloys – a computational micromechanics approach	R.S. KUMAR, A.-J. WANG and D.L. MCDOWELL	173-210

On the fracture models determined by the continuum-strong discontinuity approach	A.E. HUESPE, J. OLIVER, M.D.G. PULIDO, S. BLANCO and D. LINERO	211–229
On fractals and size effects	V.E. SAOUMA and G. FAVA	231–249
Stress trajectories for mode I and mode II cracks	J. WEERTMAN	251–259
Hashin–Shtrikman based FE analysis of the elastic behaviour of finite random composite bodies	R. LUCIANO and J.R. WILLIS	261–273
Anti-plane shear cracks approaching a bi-material interface	G.J. DVORAK and A.P. SUVOROV	275–294

*Available
online*

www.springerlink.com

Preface



Photograph by: W.F. Pfeffer

Zdeněk P. Bažant, who is honored by this special issue, was born in Prague, Czech Republic, in December 1937, to the family of a professor of geotechnical engineering, married to a Ph.D. in sociology. His grandfather was a professor of structural mechanics and former university president. Early on, Zdeněk fell in love with mathematics and in 1955 became national winner of the Mathematical Olympics. With his first paper published in 1958 he won the national student research competition. On earning (with straight A's) the Civil Engineering degree from the Czech Technical University in Prague (ČVUT) in 1960, he became the fifth generation civil engineer in the Bažant family line. He then joined a state consulting firm Dopravoprojekt as a bridge engineer. He supervised construction of an arch bridge over the Vltava River at Zbraslav, and designed several large bridges, including a unique prestressed box-girder at Kořenov (with spans of horizontal curvature extreme at that time), and won (with his sister Milada and her husband Ivo Oberstein, architects) the 2nd prize among 18 entries in an anonymous public competition for a bridge over Danube in Bratislava.

In 1963, Zdeněk earned his Ph.D. in engineering mechanics from the Czechoslovak Academy of Sciences, with a dissertation on concrete creep theory, which he then expanded into a book published in Czech. Then, while conducting research at ČVUT on fiber-polymer composites, he obtained in 1966 a postgraduate diploma in physics from Charles University in Prague, and habilitated as Associate Professor (Docent) in concrete structures at ČVUT. In 1966, he was on a half-year post-doctoral fellowship at CEBTP in Paris, to work with R. L'Hermite, who introduced him to experimental materials research.

In 1967, Zdeněk and his wife Iva managed to make their way to the University of Toronto where he witnessed Kani's pioneering tests of large beams revealing surprising size effects. The following year, Boris Bresler invited Zdeněk to work with him at UC Berkeley on gas-cooled reactor structures. In the fall of 1969, Zdeněk joined Northwestern University as Associate Professor. He became Professor of Civil Engineering in 1973, and served as Director of the Center for Concrete and Geomaterials (1981–1987), which he founded. Since 1990, he has held at Northwestern the distinguished chair of W.P. Murphy Professor of Civil Engineering and Materials Science, and since 2002 simultaneously the chair of McCormick Institute Professor.

Zdeněk is generally regarded as the world leader in research on scaling in the mechanics of solids. Unlike fluids, where scaling played a key role for over a century, the scaling problems of solids were neglected until he started to attack them theoretically and experimentally in the mid 1970s. In the classical solid mechanics theories – elasticity and plasticity, all the structural responses scale simply as power laws because there is no characteristic material length. However, cohesive fracture mechanics and damage mechanics necessarily involve a material length characterizing the size of damage or fracture process zone imposed in quasibrittle materials by their inhomogeneities. This leads to intricate scaling laws for structural strength, with a transition between power laws typically spanning several orders of magnitude of structure size (Bažant, 2004). Concrete happens to be an archetypical quasibrittle material for which the size effects are most conspicuous on the normal scale of usage.

Benefiting from his early experience with concrete, Zdeněk set forth in the early 1970s to formulate a theory explaining the complex size effects and distributed cracking observed on quasibrittle materials. In 1976, he demonstrated the spurious size effects and mesh sensitivity of strain softening models, manifested by the lack of objectivity of finite elements programs (Bažant, 1976). As a remedy, he proposed the crack band model, in which the post-peak strain softening is adjusted according to crack band width so as to guarantee correct and mesh-independent energy dissipation. Working with L. Cedolin and B.H. Oh during 1977–1983, he refined and verified this model (Bažant and Oh, 1983), which soon became a standard tool in finite element programs and commercial software for concrete design. He then embarked on exploiting the concepts of Eringen and Mindlin to prevent spurious localization of softening damage. In 1984, he proposed softening models with material characteristic length, of nonlocal (with Chang and Belytschko) and gradient type, refined later with G. Pijaudier-Cabot, F.B. Lin and M. Jirásek to a nonlocal damage or smeared cracking models, which became standard reference and stimulated further advances (Pijaudier-Cabot and Bažant, 1987; Bažant and Lin, 1988; Bažant and Jirásek, 2002). Later, with Pijaudier-Cabot, he showed how to measure the nonlocal characteristic material length (Bažant and Pijaudier-Cabot, 1989) and, with Prat, justified the nonlocal softening damage by micromechanics of interacting growing crack systems in heterogeneous materials (Prat and Bažant, 1997).

Until the mid 1980s, all the experimentally observed size effects in solids were automatically attributed to Weibull's statistical theory of random material strength. Noting that experiments on concrete could not be interpreted in this manner, Zdeněk advanced the scaling theory with his 1984 energetic (deterministic) scaling law (Bažant, 1984). This law, bridging the power scaling laws of classical fracture mechanics and plasticity, was derived by asymptotic matching of the release of stored energy due to stable

growth of large fractures or large damage zones prior to failure (Bažant and Planas, 1998; Bažant and Kazemi, 1990; Bažant, 2002, 2004; Bažant and Yavari, 2005), and has been used widely. With P. Pfeiffer, J.K. Kim, G. Pijaudier-Cabot, R. Gettu, P. Prat, M. Kazemi, M. Jirásek, J. Planas, I.M. Daniel, Z. Li, G. Zi, J.J. Kim, E. Becq-Giraudon, G. Cusatis, D. McClung, M. Brocca and Y. Zhou, he verified his law experimentally and by computer simulations for concrete, rocks, sea ice, consolidated snow and various advanced materials – fiber composites, rigid foams, sandwich systems, tough ceramics (Bažant and Kazemi, 1990; Bažant et al., 1993, 1996; Bažant and Planas, 1998; Bažant et al., 1999; Bažant, 2002; Bažant et al., 2003a, 2003b, 2004, 2006) – showed how to exploit the scaling law for measuring cohesive fracture characteristics and continuous damage parameters (Bažant and Kazemi, 1990; Bažant and Planas, 1998; Bažant, 2002); and extended his size effect law to compression fracture, including kink band propagation in fiber composites (Bažant et al., 1999; Bažant, 2002). He developed (with Y. Xi and D. Novák) a nonlocal generalization of Weibull statistical theory for strength of heterogeneous quasibrittle materials (Bažant and Xi, 1991; Bažant and Novák, 2000). Recently he justified it on the basis of stress dependence of activation energy of interatomic bonds, and showed that the probability distribution of strength gradually changes from Gaussian to Weibull as the structure size increases, which is a crucial point for ensuring tolerable failure probability of structures such as small as 10^{-6} . He solved the size effect in flexural fracture of sea ice due to temperature changes and vertical loads (Bažant, 2000a, 2002b; Bažant and Pang, 2006), and with Y.N. Li, showed how to convert the size effect problem of cohesive fracture to an eigenvalue problem with the size as the eigenvalue (Bažant and Planas, 1998). Recently, with Z. Guo and H. Espinosa, he used again asymptotic arguments to model the size effect in thin films and improve the strain-gradient theory of metal plasticity on approach to nano-scale (Bažant, 2002b; Bažant et al., 2005).

Although Zdeněk is best known for his results on scaling and nonlocal damage, he has made many other significant contributions. For creep and shrinkage of aging concrete at variable humidity and temperature, he developed, beginning with his dissertation (Bažant, 1976) in 1963, a realistic, thermodynamically based constitutive model (Bažant, 1972b) and exponential algorithm, expanded with S. Wu and J.C. Chern (Jirásek and Bažant, 2002), and conceived, with S. Baweja and A. Hauggaard, the microprestress-solidification theory (Bažant et al., 1997), recently extended with G. Cusatis. His models for nonlinear water diffusion and hygrothermal effects in non-saturated concrete (Bažant and Najjar, 1972; Bažant and Thonguthai, 1978), developed with L. Najjar and W. Thonguthai, have become standard. He is credited with the age-adjusted effective modulus method (Bažant, 1972a) for structural creep analysis, used widely in practice and featured in design codes (Jirásek and Bažant, 2002). He developed with S. Baweja widely applied creep prediction model B3 (Jirásek and Bažant, 2002), formulated with A. Steffens a chemo-mechanical fracture model for expansive alkali-silica reaction in concrete, and worked with F.J. Ulm and O. Coussy to explain massive spalling in ‘Chunnel’ fire. His microplane constitutive models (Bažant and Oh, 1985; Bažant et al., 2000) for concrete and rock (produced with B.H. Oh, P. Gambarova, P. Prat, I. Carol, Y. Xiang, F. Caner, M. Jirásek, M. Brocca, G. DiLuzio, and G. Zi) led to superior predictions of missile penetration, blast and groundshock effects, and appear in commercial software. To measure constitutive law for very large shear strains at extreme pressures, he developed the

tube-squash test, and later used this test (with F. Caner and J. Červenka) to identify the minimum confining reinforcement needed to suppress strain softening and size effect in concrete. With F. Caner, he experimentally demonstrated and modeled the vertex effect in strain-softening concrete. To predict random scatter of aging creep, shrinkage, with random environment effects, he developed (with J.C. Chern, K.L. Liu, T.S. Wang and T. Tsubaki) effective statistical sampling and spectral techniques, enhanced with Bayesian probabilistic approach (Bažant and Chern, 1984).

In the theory of structural stability, Zdeněk resolved the controversy and mutual relationship between Engesser's and Haringx's theories of shear buckling and identified proper correlations among stability criteria and objective stress rates associated with different choices of finite strain measure (Bažant, 1971; Bažant and Cedolin, 1991). Then, with A. Beghini, he clarified the finite-strain aspects of bifurcation predictions for homogenized structures very soft in shear (Bažant and Beghini, 2005). Using Eshelby theorem, he solved the conditions for damage localization into ellipsoidal domains (Bažant and Cedolin, 1991), formulated general thermodynamic criteria of stable inelastic post-bifurcation path (Bažant and Cedolin, 1991), showed that no-tension design of concrete or rock structures is not always safe (Bažant and Planas, 1998), and obtained the bifurcation criterion for arrest of alternating cracks in a parallel system of cooling or shrinkage cracks (Bažant and Cedolin, 1991). In 1972, he presented (with M. Christensen) a micropolar continuum approximation and closed-form critical load solutions for regular lattices under initial stress (Bažant and Cedolin, 1991). He found (with B.H. Oh) new more efficient Gaussian integration formulae for spherical surface (Bažant and Oh, 1986), and presented precise analysis of the problem of spurious high-frequency wave reflections due to nonuniform finite element size. He discovered and modeled (with W.H. Gu and K. Faber) the softening-hardening reversal in fracture at sudden loading rate increase. He pioneered plastic-fracturing and frictional-dilatant constitutive models for concrete (Jirásek and Bažant, 2002). For seismic sand liquefaction, he contributed (with V. Cuellar, R. Krizek and R. Blázquez) one of the earliest comprehensive models. He showed the undesirable safety consequences of hidden size effect and covert understrength factors implied in concrete design codes. In the theory of elasticity, he solved the three-dimensional singularities of skew cracks and corners in elasticity and potential theory (Bažant and Estenssoro, 1979). His advanced comprehensive treatises on stability (Bažant and Cedolin, 1991), plasticity and creep (Jirásek and Bažant, 2002), cohesive fracture (Bažant and Planas, 1998) and scaling (Bažant, 2002b) have become popular sources.

On 9/11, Zdeněk watched on the TV the collapse of WTC and saw an adjacent building, in which his daughter worked, disappear in smoke. Being an Illinois Registered Structural Engineer, he immediately began to analyze the collapse and, working with Y. Zhou, he submitted within three days to ASCE a simple mechanical analysis of the collapse, which was later translated into seven languages.

Zdeněk is a member of the National Academy of Sciences (since 2002) and National Academy of Engineering (since 1996), as one of only 153 dual members. He was also inducted to Academia di Scienze e Lettere (Italy, 2002), Austrian Academy of Sciences (2000) and Academy of Engrg. of Czech Republic (1998). He holds six honorary doctorates (TU Prague 1991, TU Karlsruhe 1997, UC Boulder 2000, Politecnico di Milano 2001, INSA Lyon 2004, and TU Vienna 2005). His honors include SES Prager Medal; ASME Warner Medal; ASCE von Kármán Medal, Newmark Medal,

Lifetime Achievement Award, Croes Medal, Huber Prize and T.Y. Lin Award; RILEM L'Hermite Medal; Am. Ceramic Soc. Roy Award; Torroja Medal (Spain); Šolín and Stodola Medals (Czech Rep., Slovakia); ICOSSAR Lecture Award; Medal of Merit (ČVUT Prague); SEA01 Meritorious Paper; Best Engrg. Book of the Year (SAP); Guggenheim, Humboldt, NATO, JSPS, Kajima and Ford Foundation Fellowships; and China Government Lectureship (Taiwan). An author of six books and over 450 papers in refereed journals*, he is an ISI Highly Cited Scientist in Engineering (ranking him among 250 most cited authors in all engineering fields worldwide; www.ISIhighlycited.com). He served as Editor of ASCE J. of Engrg. Mechanics (1988-1994) and is Regional Editor of Int. J. of Fracture (since 1991); was president of Soc. of Engineering Science (1983); founding president of IA-FRAMCOS (1991-1993) and of IA-CONCREEP (2001); Division Director in IA-SMiRT (1981-1994); and member of US Nat. Comm. on Theor. & Appl. Mech. (2000-2003).

Zdeněk is a man of broad-ranging interests. A regular patron of Chicago Symphony, he relaxes at home by playing his Steinway piano. He has been a mountain hiker and avid tennis player ever since his boyhood. In his forties, he joined Evanston Running Club, and picked up windsurfing on Lake Michigan. But the main passion of Zdeněk and his wife, a physician, has been downhill skiing. They trained their two children, Martin (now an associate professor at M.I.T.) and Eva (currently a doctoral student at Johns Hopkins) as superb skiers, and currently instruct their 13-year-old grandson Z. Steve Bažant. As a part-time ski instructor, injured on a race course, Zdeněk invented in 1959 a safety ski binding, and obtained one of the first patents on such devices. He succeeded in getting state company Lověna in Prague to produce 30,000 pairs. These bindings, branded as ZPB binding (exhibited in New England Ski Museum in Franconia, NH), were used in the early 1960s by about a third of skiers in his native country. This earned him appointment as equipment expert on the Czechoslovak State Commission for Skiing.

In 1976, Zdeněk was named Outstanding New Citizen by the Chicago Metropolitan Citizenship Council. He is enormously proud of this award, and regards his move to the US as the best decision of his career, especially for the benefit of the stimulating academic environment, without which, he feels, he would have never received the honor of this special issue.

George J. Dvorak
William Howard Hart Professor of Mechanics
Department of Mechanical, Aerospace and Nuclear Engineering,
Rensselaer Polytechnic Institute,
Troy, NY 12180-3590, U.S.A.
E-mail: dvorak@rpi.edu

*Download Bažant's papers (and book errata) from: www.civil.northwestern.edu/people/bazant.html

References

- Bažant, Z.P. (1971). A correlation study of incremental deformations and stability of continuous bodies. *Journal of Applied Mechanics Transactions of the ASME*, **38**, 919–928.
- Bažant, Z.P. (1972a). Prediction of concrete creep effects using age-adjusted effective modulus method. *American Concrete Institute Journal* **69**, 212–217.
- Bažant, Z.P. (1972b). Thermodynamics of interacting continua with surfaces and creep analysis of concrete structures. *Nuclear Engineering and Design* **20**, 477–505.
- Bažant, Z.P. (1976). Instability, ductility, and size effect in strain-softening concrete. *Journal of Engineering Mechanics – ASCE* **102**, EM2, 331–344.
- Bažant, Z.P. (1984). Size effect in blunt fracture: concrete, rock, metal. *Journal of Engineering Mechanics – ASCE*, **110**(4), 518–535.
- Bažant, Z.P. (2002a). Scaling of sea ice fracture-Part I: Vertical penetration. *Journal of Applied Mechanics – ASME* **69** (Jan.), 11–18.
- Bažant, Z.P. (2002b). *Scaling of Structural Strength*. Hermes Penton Science (Kogan Page Science): London; 2nd updated ed., Elsevier 2005 (Errata: www.civil.northwestern.edu/people/Bazant.html).
- Bažant, Z.P. (2004). Scaling theory for quasibrittle structural failure (inaugural article). *Proceedings of the National Academy of Sciences* **101**(37), 14000–14007.
- Bažant, Z.P. and Beghini, A. (2005). Which formulation allows using a constant shear modulus for small strain-buckling of soft-core sandwich structures? *Journal of Applied Mechanics ASME* **72** (Sept.), 785–787.
- Bažant, Z.P. and Cedolin, L. (1991). *Stability of Structures: Elastic, Inelastic, Fracture and Damage Theories*, Oxford University Press: New York; 2nd ed. Dover Publishers: New York (2003).
- Bažant, Z.P. and Chern, J.-C. (1984). Bayesian statistical prediction of concrete creep and shrinkage. *American Concrete Institute Journal* **81**, 319–330.
- Bažant, Z.P. and Estenssoro, L.F. (1979). Surface singularity and crack propagation. *International Journal of Solids and Structures* **15**, 405–426.
- Bažant, Z.P. and Jirásek, M. (2002). Nonlocal integral formulations of plasticity and damage: survey of progress. *Journal of Engineering Mechanics – ASCE* **128**(11), 1119–1149.
- Bažant, Z.P. and Kazemi, M. T. (1990). Determination of fracture energy, process zone length and brittleness number from size effect, with application to rock and concrete. *International Journal of Fracture*, **44**, 111–131.
- Bažant, Z.P. and Lin, F.-B. (1988). Nonlocal smeared cracking model for concrete fracture. *Journal of Structural Engineering – ASCE* **114**(11), 2493–2510.
- Bažant, Z.P. and Najjar, L. J. (1972). Nonlinear water diffusion in nonsaturated concrete. *Materials and Structures (RILEM, Paris)*, **5**, 3–20.
- Bažant, Z.P. and Novák, D. (2000). Probabilistic nonlocal theory for quasibrittle fracture initiation and size effect. I. Theory. *Journal of Engineering Mechanics – ASCE* **126**(2), 166–174.
- Bažant, Z.P. and Oh, B.H. (1983). Crack band theory for fracture of concrete. *Materials and Structures (RILEM)* **16**, 155–177.
- Bažant, Z.P. and Oh, B.-H. (1985). Microplane model for progressive fracture of concrete and rock. *Journal of the Engineering Mechanics – ASCE* **111**, 559–582.
- Bažant, Z.P. and Oh, B.-H. (1986). Efficient numerical integration on the surface of a sphere. *Zeitschrift für angewandte Mathematik und Mechanik (ZAMM)* **66**(1), 37–49.
- Bažant, Z.P. and Pang, S.D. (2006). Activation energy based extreme value statistics and size effect in brittle and quasibrittle fracture. Submitted to *International Journal of the Mechanics and Physics of Solids*.
- Bažant, Z.P. and Pijaudier-Cabot, G. (1989). Measurement of characteristic length of nonlocal continuum. *Journal of Engineering Mechanics – ASCE* **115**(4), 755–767.
- Bažant, Z.P. and Planas, J. (1998). *Fracture and Size Effect in Concrete and Other Quasibrittle Materials*. CRC Press: Boca Raton, London.
- Bažant, Z.P. and Thonguthai, W. (1978). Pore pressure and drying of concrete at high temperature. *Journal of the Engineering Materials Division – ASME* **104**, 1058–1080.
- Bažant, Z.P. and Xi, Y. (1991). Statistical size effect in quasi-brittle structures: II. nonlocal theory. *Journal of Engineering Mechanics – ASCE* **117**(11), 2623–2640.

- Bažant, Z.P. and Yavari, A. (2005). Is the cause of size effect on structural strength fractal or energetic-statistical? *Engineering Fracture Mechanics* **72**, 1–31.
- Bažant, Z.P., Caner, F.C., Carol, I., Adley, M.D. and Akers, S.A. (2000). Microplane model M4 for concrete: I. formulation with work-conjugate deviatoric stress. *Journal of the Engineering Mechanics – ASCE* **126**(9), 944–953.
- Bažant, Z.P., Daniel, I.M. and Li, Z. (1996). Size effect and fracture characteristics of composite laminates. *Journal of Engineering Materials and Technology – Transactions of the ASME* **118**(3), 317–324.
- Bažant, Z.P., Guo, Z., Espinosa, H., Zhu, Y. and Peng, B. (2005). Epitaxially influenced boundary layer model for size effect in thin metallic films. *Journal of Applied Physics* **97**, 073506–1–073506–13.
- Bažant, Z.P., Hauggaard, A.B., Baweja, S. and Ulm, F.-J. (1997). Microprestress-solidification theory for concrete creep. I. aging and drying effects, *Journal of Engineering Mechanics – ASCE* **123**(11), 1188–1194.
- Bažant, Z.P., Kim, J.J. H., Daniel, I.M., Becq-Giraudon, E. and Zi, G. (1999). Size effect on compression strength of fiber composites failing by kink band propagation. *International Journal of Fracture* **95**, 103–141.
- Bažant, Z.P., Lin, F.-B. and Lippmann, H. (1993). Fracture energy release and size effect in borehole breakout. *International Journal for Numerical and Analytical Methods in Geomechanics* **17**, 1–14.
- Bažant, Z.P. Zhou, Y. and Daniel, I.M. (2006). Size effect on strength of laminate-foam sandwich plates, *Journal of Engineering Materials and Technology – Transactions of the ASME* **128**, in press.
- Bažant, Z.P., Zhou, Y., Novák, D. and Daniel, I.M. (2004). Size effect on flexural strength of fiber-composite laminate. *Journal of Engineering Materials and Technology – Transactions of the ASME* **126** (Jan.), 29–37.
- Bažant, Z.P., Zhou, Y., Zi, G. and Daniel, I.M. (2003a). Size effect and asymptotic matching analysis of fracture of closed-cell polymeric foam. *International Journal of Solids and Structures* **40**, 7197–7217.
- Bažant, Z.P., Zi, G. and McClung, D. (2003b). Size effect law and fracture mechanics of the triggering of dry snow slab avalanches. *Journal of Geophysical Research* **108** (B2), 2119–2129.
- Jirásek, M. and Bažant, Z.P. (2002). *Inelastic Analysis of Structures*. John Wiley & Sons: London, New York.
- Pijaudier-Cabot, G. and Bažant, Z.P. (1987). Nonlocal damage theory. *Journal of Engineering Mechanics – ASCE* **113**(10), 1512–1533.
- Prat, P.C. and Bažant, Z.P. (1997). Tangential stiffness of elastic materials with systems of growing or closing cracks. *Journal of the Mechanics and Physics of Solids* **45**(4), 611–636 and **45**(8), 1419–1420.