

# Gap Test of Crack-Parallel Stress Effect on Quasibrittle Fracture and Its Consequences

**Hoang Thai Nguyen**

Graduate Research Assistant  
Theoretical and Applied Mechanics program and  
Department of Civil & Environmental Engineering  
Northwestern University  
Evanston, Illinois 60208

**Madura Pathirage**

Graduate Research Assistant  
Department of Civil & Environmental Engineering  
Northwestern University  
Evanston, Illinois 60208

**Gianluca Cusatis**

Professor of Civil and Environmental Engineering  
Department of Civil & Environmental Engineering  
Northwestern University  
Evanston, Illinois 60208

**Zdeněk P. Bažant \***

Walter P. Murphy Professor of Civil and Environmental Engineering,  
Mechanical Engineering and Material Science and Engineering  
Department of Civil & Environmental Engineering  
Northwestern University  
Evanston, Illinois 60208  
Email: z-bazant@northwestern.edu

## ABSTRACT

In the standard fracture test specimens, the crack-parallel normal stress is negligible. However, its effect can be strong, as revealed by a new type experiment, briefly named the *gap test*. It consists of a simple modification of the standard three-point-bend test whose main idea is to use plastic pads with a near-perfect yield plateau to generate a constant crack-parallel compression, and install

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\*Address all correspondence to this author.

the end supports with a gap that closes only when the pads yield. This way, the test beam transits from one statically determinate loading configuration to another, making evaluation unambiguous. For concrete, the gap test showed that moderate crack-parallel compressive stress can increase up to 1.8 times the Mode I (opening) fracture energy of concrete, and reduce it to almost zero on approach to the compressive stress limit. To model it, the fracture process zone must be characterized tensorially. Here we use computer simulations with the crack-band microplane model, considering both in-plane and out-of-plane crack-parallel stresses for plain and fiber-reinforced concretes. The results have broad implications for all quasibrittle materials, e.g., shale, fiber composites, sea ice, foams, and bone. Except when the crack-parallel stress is negligible, the line crack models are shown to be inapplicable. Nevertheless, as an approximation ignoring stress tensor history, the crack-parallel stress effect may be introduced parametrically, by a simple formula. Finally we show that the standard tensorial strength models such as Drucker-Prager cannot reproduce these effects realistically.

**Key Words:** Fracture testing, fracture mechanics, fracture properties, finite element analysis, concrete, fiber reinforced materials, computational mechanics

## 1 INTRODUCTION

In a preceding article, [1] presented a simple novel experiment, named for brevity the *gap test*, which showed that a moderate crack-parallel compressive stress can greatly increase the Mode-I fracture energy,  $G_f$ , of concrete, and a high stress can reduce it almost to zero. This surprising result, with its mesomechanical explanation in [1], suggests major consequences for the numerical simulations of fracture—not only for concrete but also for all quasibrittle materials. Why hasn't this phenomenon been identified before? Three reasons can be seen:

1) All the standard notched fracture specimens (three-point-bend, compact tension, single edge-notched tension, eccentric compression, etc.) have a negligible normal stress parallel to the crack. This is nearly so even for the standard wedge-splitting specimen.

2) Fracture analysis has generally been focused on line cracks, as in linear elastic fracture mechanics (LEFM), originated in 1921 Griffith [2], or in cohesive crack model (CCM), pioneered in 1959 by Barenblatt [3,4].

3) Visually, cracks appear as lines (albeit rough), and it is obvious that a line cut in direction  $x$  in a

specimen under homogeneous uniaxial stress  $\sigma_{xx}$  causes no stress change, which indicates that  $\sigma_{xx}$  does not matter for a continuum model.

It is thus clear that the crack-parallel stress can matter only if the fracture process zone (FPZ) at crack front has a finite width,  $w_c$ . This is known to be a property of all quasibrittle materials, which are heterogeneous materials consisting of brittle constituents and inhomogeneities not negligible compared the structure dimensions. These materials include concrete, as the archetypical case, fiber reinforced concrete, shale and various rocks, fiber-polymer composites, coarse-grained or toughened ceramics, bone and many bio-materials, stiff soils, sea ice, various printed materials, rigid foams, wood, etc., and all brittle materials on the micrometer scale.

When short concrete specimens under homogeneous uniaxial compression are about to fail by axial splitting cracks [5], the fracture energy for a superposed loading in the opening mode I would clearly be zero [6–13].

This suggests that, in concrete and similar quasibrittle materials, the  $G_f$  should depend on crack-parallel stress  $\sigma_{xx}$  and should terminate with 0 when  $\sigma_{xx}$  tends to uniaxial compressive stress limit  $\sigma_c$ . This represents the value of crack-parallel stress that causes compressive failure in the three-point-bend specimen. Its value is slightly different from  $f_c$  (uniaxial compressive stress of  $8 \times 4$  in. cylinders) due to the differences in specimen geometry and boundary conditions.

The CCM [14, e.g.] cannot capture the effect of crack-parallel stress  $\sigma_{xx}$  because it is a line crack model, in which the crack-parallel stress or strain is not the basic thermodynamic variable. It can be introduced only as parameter, but then the dependence of  $G_f$  on the stress tensor history is missed, as documented here by an example (this is similar to the inadequacy of describing plasticity in terms of total, rather than incremental, stress-strain relations).

On the other hand, the crack band model [14–16] (CBM), with a tensorial softening damage constitutive law, can serve this purpose, provided that the damage law used is sufficiently realistic, i.e., the constitutive damage law must capture not only tensile microcracking in the FPZ, but also the frictional slip on inclined planes (see Fig. 2 in [1]) accompanied by splitting microcracks, which dissipate more energy than the tensile microcracking and is what causes the lateral expansion and longitudinal shortening of the FPZ under crack-parallel compression [17]. On the constitutive macro-level, these mechanisms cause a strong vertex effect [18–20] representing, e.g., a drastically reduced incremental stiffness for shear superposed on inelastic compression.

The microplane model for concrete used in the CBM, particularly its latest version M7 [21], has been

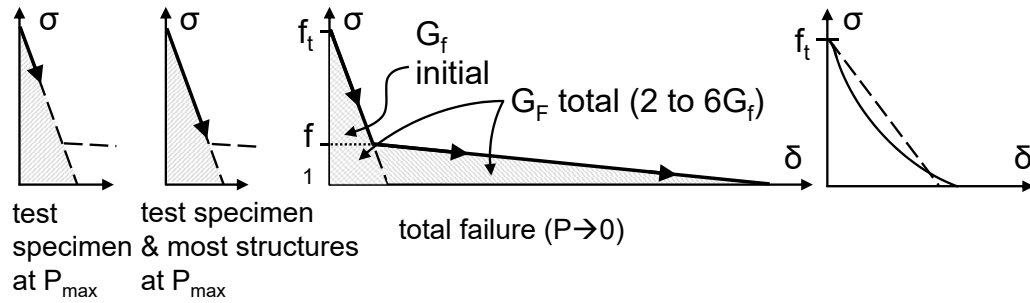


Fig. 1. Traction-separation curve without crack-parallel stress.

shown to reproduce well the compression fracture as well as the measured crack-parallel stress effect on concrete [1]. M7 is adopted here to explore various interesting consequences of the new gap test. One feature of the microplane model that is important for the present problem is that it can reproduce the vertex effect and capture separately the frictional slip of microcracks on planes with various distinct orientations (note that the tensorially defined constitutive laws which model internal friction by interaction between the first and second invariants can make no distinction among various orientations of frictional slip). Importantly, by means of its deviatoric stress boundary, M7 automatically captures the effect of the spreading and splitting microcracks.

In discussing the fracture energy, we must first clarify what kind. The typical softening stress-separation curve of CCM of concrete begins with a steep initial tangent, followed by a very long tail, as shown in the last diagram in Fig. 1. This originally used curve was later found to lead to size-dependent  $G_F$ . Accordingly, two fracture energies are distinguished: (a) the initial fracture energy,  $G_f$ , given by the area under the initial tangent, and the total fracture energy,  $G_F$ , representing the area under the complete softening curve, including its tail [14, 22–24]. Typically,  $G_F/G_f \approx 2$  to 6 for concretes. The  $G_F$  suffers from high uncertainty unless the work-of-fracture tests are conducted at various specimen sizes [25]. The  $G_f$  alone governs the load capacity of most structures, while the  $G_F$  governs the energy absorption in dynamic failures such as impact. Here we consider the  $G_f$  only, from which the  $G_F$  can be inferred approximately (Fig. 1 shows also on right softening curves without tail which were introduced around 1980 by Hillerborg and others).

Unambiguous determination of  $G_f$  (and characteristic FPZ size  $c_f$ ) from the present experiments and numerical simulations is made possible by the size effect method (see Appendix E). This method [14, 26], embodied in the international standard recommendation of RILEM [27], endorsed recently by ACI-446 committee, and improved in [23, 24, 28], was devised in 1990 (and without  $c_f$  in 1987 [22]). It is an effective way to circumvent the impossibility of determining, optically or acoustically, the precise location of the tip

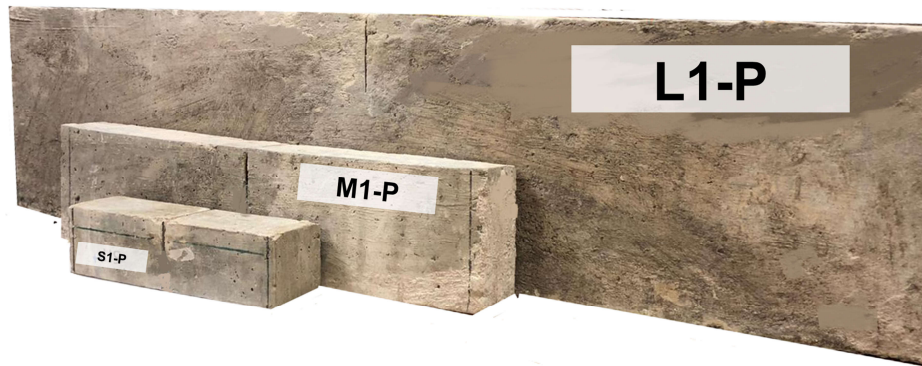


Fig. 2. Concrete test specimens scaled geometrically in two dimensions.

of a macroscopically equivalent LEFM crack within the large FPZ of quasibrittle materials. In this method, based on asymptotic matching, it suffices to measure the maximum loads,  $P$ , of notched specimens of several sufficiently different sizes, preferably, but not necessarily, geometrically similar. The geometrically similar notched concrete beams of three sizes shown in Fig. 2 have been used here.

The preceding computer simulations [1] of the gap test dealt with only with a few cases, with only a few levels of the crack-parallel stress, and only with concrete. These simulations demonstrated that the concrete fracture energy values measured by the gap test at several different crack-parallel stress levels are closely matched by the predictions of the finite element (FE) CBM with the microplane model M7 as the constitutive damage law [21, 29]. Here we continue to use the CBM with M7 to demonstrate and clarify the effect of crack-parallel stresses in many more situations. Furthermore, using the microplane model M7f [30] and the anisotropic spherocylindrical microplane model [31], we explore the crack-parallel stress effects in two other quasibrittle materials—the fiber reinforced concrete, and shale.

From the practical viewpoint, this study has been motivated by the impossibility to match with the CCM the test results for shear failure of reinforced concrete beams and slabs; ditto for the splitting fracture of a sea ice plate moving against the legs of an oil platform, a vertical pavement crack under wheel load, a hydraulic crack in shale, etc.

## 2 OVERVIEW OF BASIC ASPECTS OF THE NOVEL EXPERIMENT—GAP TEST

The novel, yet surprisingly simple, gap test [1], depicted in Fig. 3a, has four key features:

- 1) A pair of compressible pads (Fig. 3a,b) capable of nearly perfect plastic yielding (Fig. 3c) is installed next to the notch mouth to produce notch-parallel compression of desired magnitude (Fig. 3c),

with no bending moment (Fig. 4a-i).

2) Rigid supports at beam ends are installed with a gap (of about 2 to 4 mm) so as to engage in contact and apply the crack-producing bending moment (Fig. 3a,b) only after the pads start plastic yielding (Fig. 4a-ii,iii).

3) This way the test beam passes from one statically determinate system to another, which makes evaluation simple and unambiguous.

4) The static determinacy of loading and the constancy of crack-parallel compression make it possible to use the size effect method, which is an easy and robust way to measure  $G_f$ , with its dependence on  $\sigma_{xx}$ .

The deflection relative to the end supports causes the load-deflection curve to rise above the yield plateau, peak and then descend back to the plateau, as seen in Fig. 4b. The area between this up-and-down curve is the energy dissipated by fracture, which is exploited in the work-of-fracture method of measuring fracture energy. However, only the peak loads, for several different specimen sizes, are needed to determine  $G_f$ ; Fig. 4c. Typical measured curves of load  $P$  versus load-point displacement  $u$ , and of  $P$  versus the crack tip opening displacement,  $\delta_{CTOD}$ , are shown for  $D = 101.6$  mm in Fig. 4b,c.

The statical determinacy of the beam when the pads are yielding is gained from the constancy of the reactions of the pads. It makes the pad reactions equivalent to applied dead loads. Hence, after the pads begin to yield, there is again three-point bending, in which the deformation of the beam cannot break symmetry of the left and right halves of the beam, even in postpeak softening. Therefore, any small initial imperfection (such as a small Mode II component) must remain small, i.e., there is no equilibrium path bifurcation. The magnitude of the yield force is controlled by the area of the pad. Smaller yield forces are obtained by drilling holes in the pads (see Appendix A).

### 3 FURTHER ASPECTS AND DETAILS OF THE NOVEL EXPERIMENT

The size effect method, a robust and simple method developed in (1990) [26], is what allows identifying  $G_f$  from the maximum load of fracture specimens of different sizes. Based on the Type 2 (energetic, non-statistical) size effect law of quasibrittle fracture [14, 26, 32–35], it has been adopted as an international standard recommendation [27], and endorsed by ACI-446 [36]. It has become the most widely used method for testing  $G_f$  of concrete and geomaterials. One advantage is that it necessitates measuring only the maximum loads,  $P_{max}$ , postpeak being superfluous, but  $P_{max}$  must be measured for at least three sufficiently different specimen sizes [26] (preferably, but not necessarily [14] scaled geometrically). As another advantage, the identification of  $G_f$ , along with the material characteristic length  $c_f$ , is reducible to linear

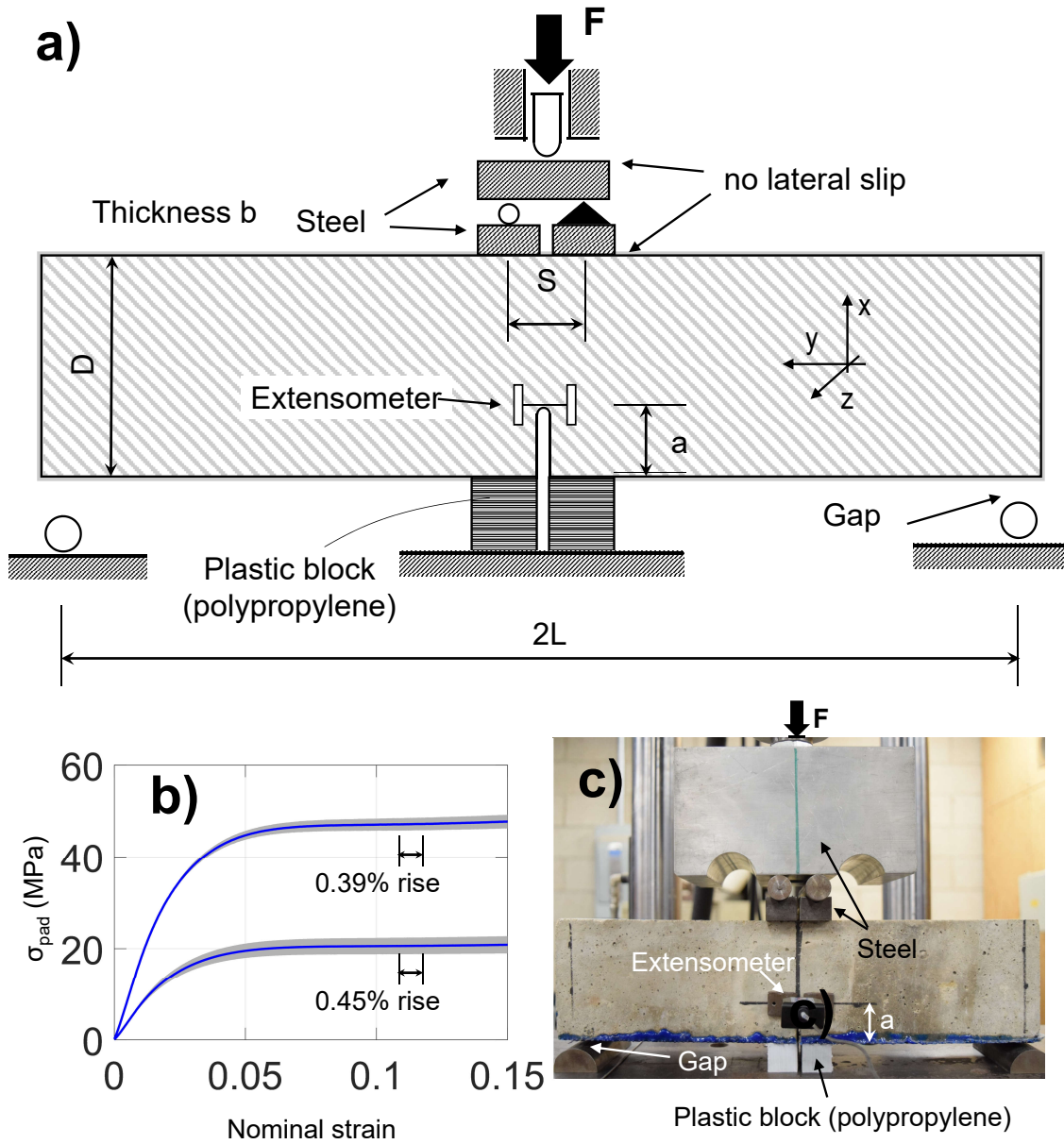


Fig. 3. (a) Schematic experimental set-up (with coordinates  $x$ ,  $y$ ,  $z$ ); (b) Real set-up; and (c) Stress-strain behavior of plastic pad corresponding to two values of tested  $\sigma_{xx}$  (note that only a short segment of the quasi-plateau intervened during the rise of bending moment, as marked in Fig. 4b).

regression. Importantly, the derivation of this method [14, 26] is not affected by the crack-parallel stress, neither in-plane  $\sigma_{xx}$  nor anti-plane  $\sigma_{zz}$ .

The test specimens are analyzed as four-point-bend beams, although the two center-span loads are spaced so closely that the beams are almost equivalent to the standard three-point bend beams. To keep the evaluation simple, the separation of center span loads was scaled with the beam size  $D$ .

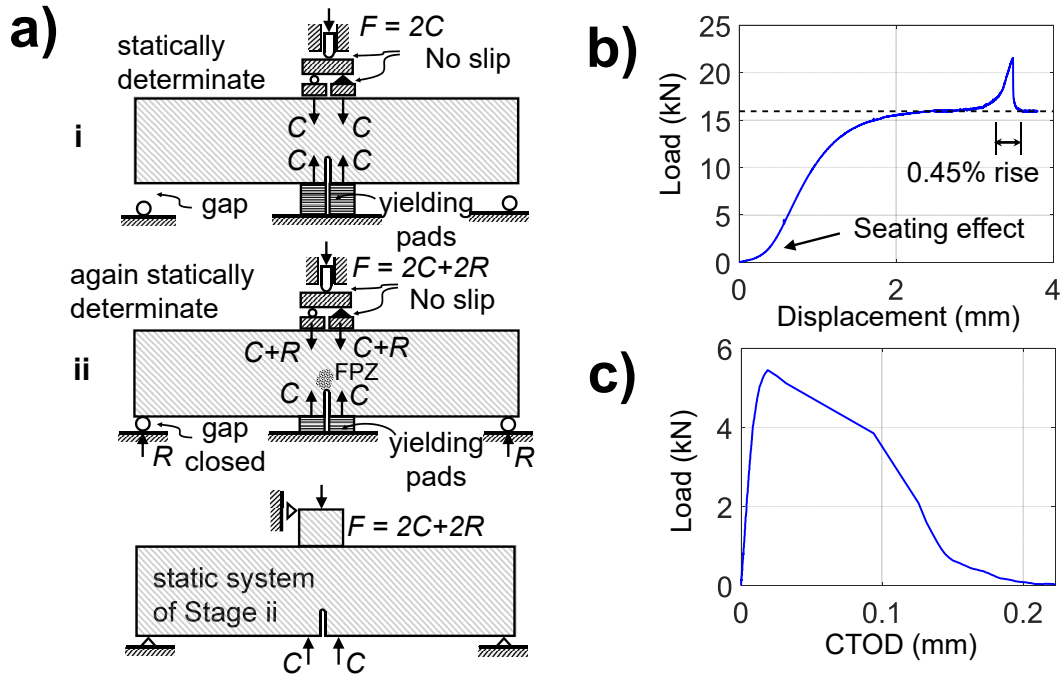


Fig. 4. (a) Experimental procedure; (b) A load-machine displacement behavior (note that within the segment  $F$ , which is what matters for  $G_f$ , the change in pad reaction (dashed line), is negligible); and (c) Extracted load-CTOD.

Beams of three depths  $D = 101.6$  mm (4 in.), 203.2 mm (8 in.) and 406.4 mm (16 in.), measured from top face to the bottom face, were tested. The span-to-depth ratio was  $2L/D = 3.75$ , and notch depth ratio  $a/D = 0.3$ . The beam thickness was 101.6 mm for all sizes. The beams were scaled geometrically in two dimensions (2D). Normal concrete with mean cylindrical compression strength  $f_c = 40.5$  MPa was used. The maximum aggregate size was  $d_a = 18$  mm. The notch width was 3 mm, which is known to be well within the admissible range compared to  $d_a$ .

The negative bending moment caused by self-weight before the pads engage is negligible (for the heaviest beam,  $< 3\%$  of the maximum bending moment). A small restraint against rotation ensures stability before the end supports engage [37–39].

The pads consist of a layer of polypropylene. Once plasticized, its tangential hardening shear modulus  $\mu$  is very small, about 20 MPa (see Fig. 3c) (it must be nonzero, or else the plastic would escape from the pads like a fluid). During plastic compression, the polymer behaves as incompressible. The compression causes the polypropylene to be laterally squeezed out, though imperceptibly so. The lateral squeezing governs the tangential hardening stiffness  $H$  of the yielding pads, which is also very small but nonzero (see Appendix E for optimization of pad shape). The pair of elasto-plastic pads is placed next to the notch mouth (Fig. 3a), and a pair of symmetric the steel loading pads on the opposite beam face. Fig. 3c documents that



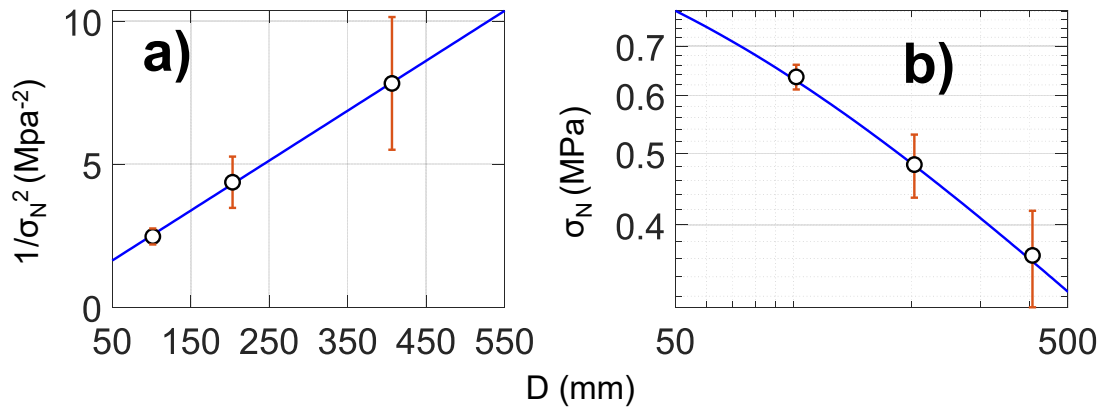


Fig. 5. (a) Linear regression of size effect method and (b) The size effect curve plotted in log-log scale.

the measured load-deflection diagram of the pads exhibits a long near-horizontal yield plateau. To lower the value of the tangential pad stiffness,  $H$  (with limited shortening of the yield plateau), regularly spaced holes are drilled through the pads (this also reduces the elastic stiffness); Fig. 3c.

To prevent shear failure of concrete under the elasto-plastic pads, a 2 mm laminate layer has been glued to the concrete surface, and it has been calculated that its effect on the stress intensity factor is negligible. The loading rate is scaled so as to reach the maximum load within approximately the same time. An extensometer crossing the crack tip is used to measure the crack tip opening displacement,  $\delta_{CTOD}$  (Fig. 3a or 4a). The complete set-up is seen in Fig. 4b.

Each of the three data points for different  $\sigma_{xx}$  has been obtained by linear regression of experiments on three sizes (see Fig. 5) and shown as the empty circles in Fig. 6a, . The coefficients of variation of the regressions (i.e., the root-mean-square of the deviations from the regression line divided by the data mean) were only 0.087%, 0.113%, 0.211%. These data points represent the effective values of fracture energy  $G_f$  as a function of three levels of compression stress  $\sigma_{pad}$  applied at the yielding pads. These experimental results [1] reveal an important fact— $G_f$  is not constant but depends on  $\sigma_{pad}$ , in fact, strongly. This raises doubts about the applicability of both the LFM and the CCM, each of which requires constancy of  $G_f$ .

To get the effective  $G_f$  as a material property,  $\sigma_{pad}$  must be transformed to the crack-parallel normal stress  $\sigma_{xx}$  at the at the notch tip. FE analysis was used to get the ratio  $r_c = \sigma_{xx}/\sigma_{pad}$ . Elastic analysis gave  $r_c = 0.962$ , while crack band finite element analysis with M7 gave  $r_c = 0.942$  for medium samples. The elastic values is seen to be a good enough approximation, with an error less than 3%. That  $\sigma_{xx}$  must be less than  $\sigma_{pad}$  is intuitively obvious from the field of principal stress vectors in Fig. 6c obtained by FE. The data points of  $\sigma_{xx}$  obtained after transformation with  $r_c$  are shown as the solid squares in Fig. 6a.

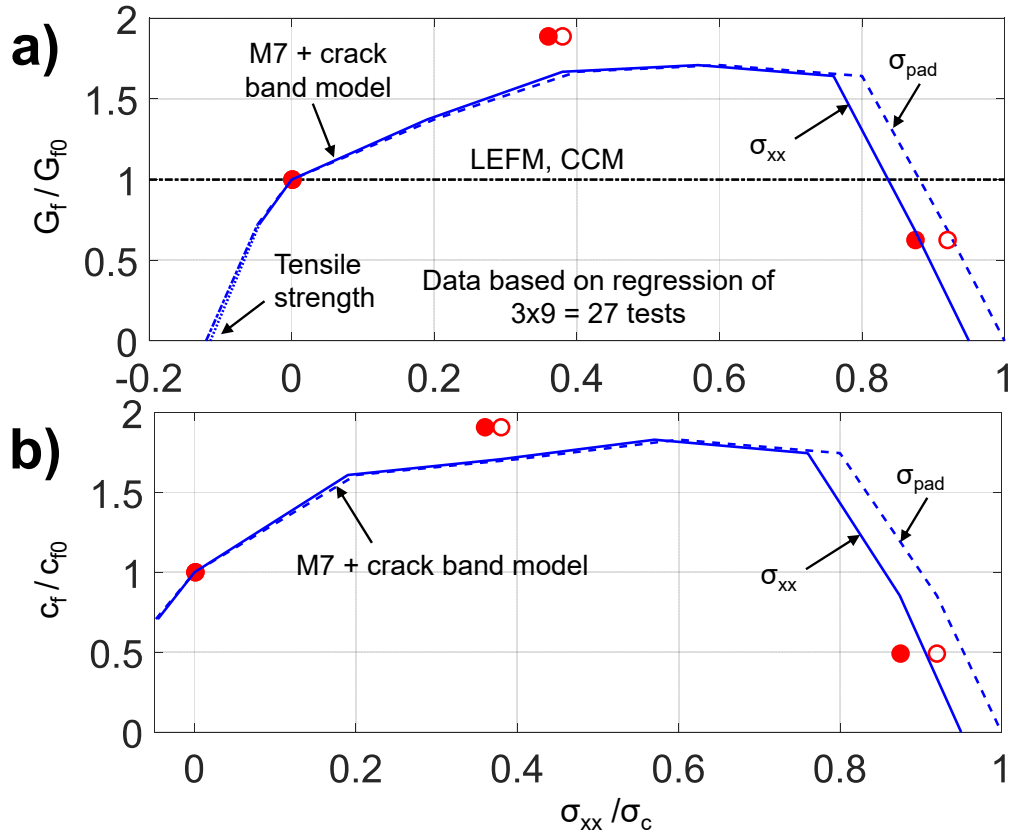


Fig. 6. (a)  $G_f$  ( $G_{f0} = 86.7$  N/m) as a function of  $\sigma_{pad}$  (dashed curve) and of  $\sigma_{xx}$  (solid curve); (b)  $c_f$  ( $c_{f0} = 8.8$  mm) as a function of  $\sigma_{pad}$  (dashed curve) and of  $\sigma_{xx}$  (solid curve).

Alternatively, according to the classical work-of-fracture method [40–42], one could estimate the total fracture energy,  $G_F$ , via the area between the entire up-and-down curve and the horizontal yield line in Fig. 4c. However, this method requires stabilizing the postpeak softening and is rather ambiguous if the correct shape of the cohesive law, Fig. 1a, is not known *a priori* [25]. To avoid ambiguity of  $G_F$ , the work-of-fracture test would have to be also conducted for several sufficiently different specimen sizes [25].

#### 4 PHYSICAL MECHANISM OF CRACK-PARALLEL STRESS EFFECT ON $G_F$

The mechanism was briefly discussed in the preceding paper [1] and here we are more specific. Depending on the magnitude of crack parallel compressive stress  $\sigma_{xx}$  (negative for compression), we can distinguish two different regimes of FPZ behavior, explained by two different mesoscale mechanisms:

*Regime 1: Friction:* Moderate  $\sigma_{xx}$  increases static friction which prevents slip, and provides confinement without damage. This tends to increase strength and may be explained by the increase of interlocking and increase of static friction on rough inclined surfaces. For  $\sigma_{xx} = 0$ , the deviatoric (or shear) stress intensity

$\tau = \tau_a$ , shown in the figure, corresponds to hydrostatic pressure is  $p = p_a = -\sigma_{yy}/3$ , where  $\sigma_{yy}$  is the tensile normal stress caused by beam bending in the FPZ at crack or notch front; see Fig. 7a ( $\tau = \sqrt{J_2}$  where  $J_2 =$  second invariant of the stress deviator). By applying a not too high compressive stress  $\sigma_{xx}$ , the hydrostatic pressure increases from  $p_a$  to  $p = p_b = (\sigma_{xx} - \sigma_{yy})/3$  and the stress state corresponds to an expanded circle. Thus  $p_b$  provides confinement, which increases the resistance  $\tau$  from  $\tau_a$  to  $\tau_b$ , as shown in Fig. 7b. Moreover, the numerical simulations show the active FPZ to become longer and narrower, which may be explained by a reduction of the average inclination of the microcracks from the macrocrack direction.

*Regime 2: Expansive Slipping and Splitting:* When, however, compression  $\sigma_{xx}$  is raised to approach the compressive stress limit, one must consider the Mohr circle shifted to the left in Fig. 7c. The minimum principal stress indicates the uniaxial splitting strength of concrete, where, FPZ is failing in compression due to  $\sigma_{xx}$ . The static friction on inclined microcracks in the FPZ is overcome and the cracks slip, which tends to widen the active FPZ and reduce the resistance to tensile stress  $\sigma_{yy}$  caused by bending, as intuitively explained by sliding over the entire FPZ portrayed in Fig. 7c. This kind of failure mechanism has been observed in simulations with the crack band microplane model, by inspecting the stresses and deformations in FPZ on microplanes of inclined orientations. Another possible mechanism is the arise and collapse of splitting micro-cracks between micropillar of width  $s$ , [17], which also leads to lateral widening (Fig. 7b) and shortening of FPZ. The widening and shortening of the FPZ in this regime is manifested in a decrease of  $c_f$  obtained from the size effect method.

The widening of the FPZ is accompanied by widening of the crack or notch behind it, which further allows the compressed material at crack faces to expand into the crack space and thus reduce the resistance to compression. For zero crack-parallel compression, the gap test gave  $G_f = 86.7$  N/m, which is within the range of values reported by many authors [14]. The characteristic length in this case is approximately 18 mm. This is about 1.5 times the average coarse aggregate size.

For a moderate crack-parallel compression,  $\sigma_{xx} \approx 0.4\sigma_c$ , the size effect analysis of the present test data yields  $G_f = 154.2$  N/m, which roughly doubles the value at zero compression (Fig. 9a).

The material characteristic length is also nearly doubled, to  $c_f = 37.1$  mm, and the FPZ gets more elongated. Another consequence is that the postpeak softening slope gets less steep or that the snapback is suppressed.

For high crack-parallel compression at  $\sigma_{xx} = 0.9\sigma_c$ , the fracture energy is, by contrast, drastically reduced—to  $G_f = 51.2$  N/m (Fig. 7a). So is the value of characteristic length,  $c_f = 11.2$  mm, which indicates a shorter and wider FPZ, and a reduced  $c_f$ . As  $D \rightarrow \infty$ , the final asymptotic slope of LEFM,  $-1/2$ ,

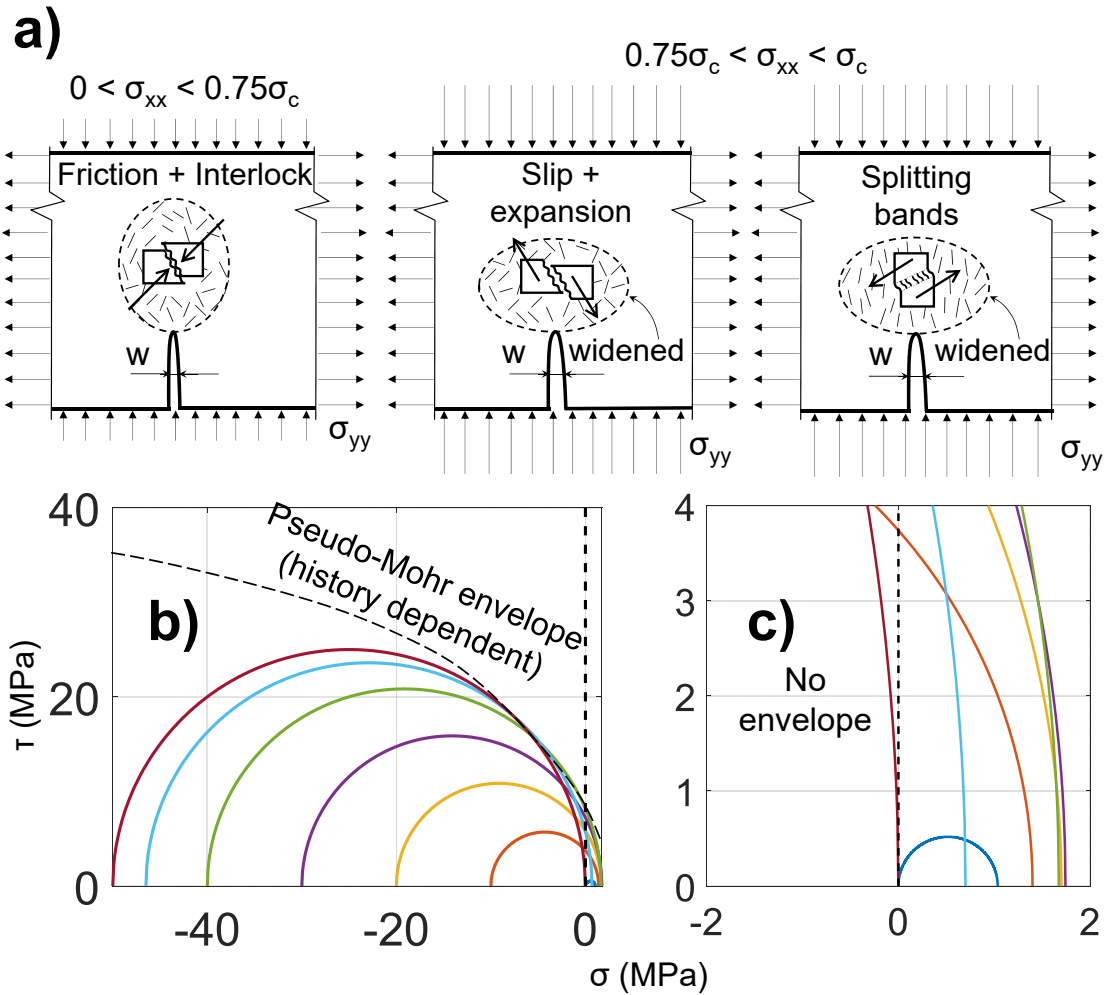


Fig. 7. (a) Suggested mechanisms for enhancement and diminishing of  $G_f$ ; (b) Mohr circles corresponding to the M7-CBM predictions in Fig. 6a), with  $\sigma_{yy}$  = nominal strength at peak load for samples of medium size; and (c) A closer look to the region of small  $\sigma_{xx}$ .

is approached more quickly. Hence, the brittleness number  $D/D_0$  [14,22] for a given  $D$  increases. After the experiments, the fractography showed small, crumbly pieces of nearly detached concrete, demonstrating the formation of small splitting microcracks and slip expansion. Another consequence is that the postpeak softening slope gets steeper or that snapback gets promoted.

## 5 FURTHER SIMULATIONS OF CRACK-PARALLEL STRESS EFFECT ON $G_F$

### Plane strain and anti-plane normal strain effects

Another interesting aspect is the effect of the antiplane stress  $\sigma_{zz}$ . For  $\sigma_{zz}$  applied on the entire side faces of the present specimens of three sizes, several simulated curves of  $G_f(\sigma_{xx})$  are plotted in Fig. 8a,b

for various ratios  $\sigma_{zz}/\sigma_c$ . Obviously, in this case (apparently never studied before),  $\sigma_{zz}$ , too, has a significant effect.

### *Fiber reinforced concrete (FRC)*

Short random fibers change significantly the post-peak softening damage in concrete. Microplane model M7f was calibrated to capture it [30] and has been used to simulate  $G_f$  for the present specimen geometry; see Fig. 8c,d. Due to the lack of material for calibration, we extrapolated the currently used model to account for the presence of fibers. First, the parameters for plain concrete were determined from calibration as described in Appendix C. Later, the parameters related to fibers were obtained from [30] as if 3% Dramix fiber was mixed to the same batch of concrete. Later, the same set of different specimen sizes were numerically tested and results were provided in Fig. 8c,d. Note that, this procedure is compliant with the calibration of M7f for FRC.

The presence of fibers enhanced the cohesive strength and the crack bridging effect, which gives a higher  $G_f$  at zero  $\sigma_{xx}$ . Therefore, a mild increase of  $\sigma_{xx}$  also showed a relative increase in  $G_f$ , but with a lower magnitude. This was probably stemming from the fact that the effect of fiber on the friction boundary is weaker than on the normal boundary of M7, i.e., the fibers are more effective in bridging the opening micro-cracks rather than preventing their sliding. Furthermore, the decline of  $G_f$  towards zero is more gradual than in plain concrete. This phenomenon originates from the inhibition of splitting micro-cracks, so that a higher  $sig_{xx}$  is needed to cause slip expansion and energy dissipation becomes more gradual (see Fig. 7).

### *Shale*

The recently developed sphero-cylindrical microplane model for shale [31], intended for the simulation of hydraulic fracturing, can capture the effect of anisotropy due to bedding layers on the damage development in shale. The orientation of the crack plane relative to bedding layers matters. Because the overburden pressure is about the double of the horizontal tectonic stresses and the bedding layers are nearly horizontal, we simulate a shale specimen of the present geometry with a crack normal to the bedding layers. For comparison, we add the case of a crack parallel to the bedding layers. The resulting curve of  $G_f$  versus  $\sigma_{xx}$  is shown in Fig. 8e,f. In the case of the crack normal to the bedding plane, the crack must cut through these layers which are stronger than the interlayer material. Thus a higher energy release is required, which explains why compressive  $\sigma_{xx}$  leads to much higher relative  $G_f$ -values than those for cracks parallel to bedding layers, as seen in Fig. 8e.

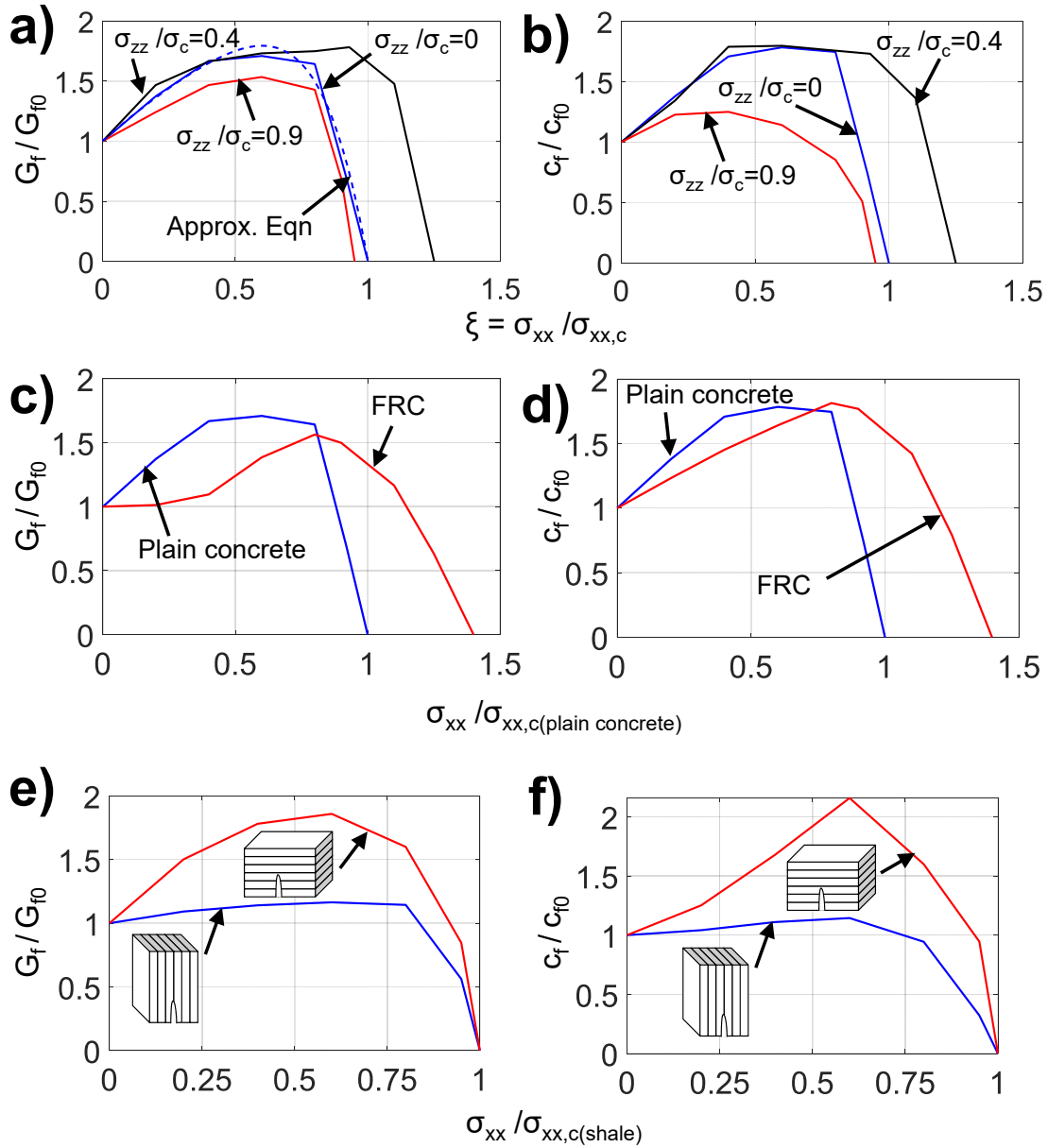


Fig. 8. (a–b)  $G_f$  and  $c_f$  as functions of  $\sigma_{xx}$  subject to different values of anti-plane stress  $\sigma_{zz}$  (with results approximated by Eqn. 1); (c–d) The variation of  $G_f$  and  $c_f$  of 3% Dramix-fiber-reinforced concrete as functions of  $\sigma_{xx}$ ; (c–d) The variation of  $G_f$  and  $c_f$  of shale with bedding plane normal and parallel to the crack as functions of  $\sigma_{xx}$ .

*History-dependence of the crack-parallel stress*

To demonstrate the path dependence, which is ignored by any formula for  $G_f$  (such as Eq. (1), Fig. 9 presents, for the mid-size concrete specimen, the M7 simulations for four different paths (or histories) of  $2R/bD$  versus  $\sigma_{xx}$ , where  $R$  is the beam end reaction that produces the bending moment,  $M$ , to which the stress-intensity factor  $K_I$  is proportional. 1) The first path is a simple increase of  $R$  at  $\sigma_{xx} = 0$  up

to failure, which defines the nominal strength  $\sigma_N$ . 2) In the gap test path, as already described, a high crack-parallel compression  $\sigma_{xx} = 0.9\sigma_c$  is applied first, and the subsequent path segment with increasing  $R$  ends up with the failure value of  $2R/bD = 0.62\sigma_N$ , as seen in Fig. 6. 3) When, however, the end reaction,  $R = 0.62\sigma_N$  is applied first (which would require a more complex test setup and controls), and the crack-parallel compression  $\sigma_{xx}$  is subsequently increased up to failure, the failure occurs at significantly higher compression  $\sigma_{xx}$  (because FPZ was less damaged during the first segment of the path. 4) When only a moderate crack-parallel compression  $\sigma_{xx} \approx 0.4\sigma_c$  is applied first, the reaction  $R$  can be increased to a much higher value, with  $2R/bD = 1.81\sigma_N$ .

### *Comparisons with tensorial plastic-failure models*

1) *Model CDPM2* – constitutive damage plastic model for concrete of the classical type, defined in terms of stress and strain tensors and their invariants. This is an excellent model that has recently been developed by Grassl *et al.* [43], within the framework of tensorial plasticity with damage. The  $G_f$  variation obtained with this model is shown in Fig. 9b. The qualitative trend obtained reflects both strengthening and weakening phases of  $G_f$  evolution, but the differences from M7 are significant and the deviations from the gap test in the last weakening phase is large.

*Drucker-Prager and Mohr-Coulomb model from Abaqus:* The  $G_f$  evolution calculated with the classical Drucker-Prager's (D-P) model [44] is presented in Fig. 9c. The strengthening phase is represented, but there is no weakening phase. For this model, the uniaxial compression stress test produces bulging at mid-length of the specimen rather than formations of splitting cracks and inclined shear bands. The strengthening phase gets captured because the D-P described well the frictional resistance to slip and interlock under triaxial confinement. It should also be mentioned that, at high compression  $\sigma_{xx}$ , the size effect method is inapplicable because there is premature compression failure at the crack front. This gets manifested as a macro-crack in front of the notch, which significantly lowers the peak load, especially in the largest specimens (Fig. 9d). This stems from the fact that the largest specimens are less confined by the boundary. Therefore, at higher compression, the size effect  $G_f$  had to be computed from linear regression of the results for only the small-size and mid-size specimens, as represented by the dashed curve segment in Fig. 9c.

*Mohr-Coulomb concrete model with a cap, from Abaqus:* Unlike the D-P, the Mohr-Coulomb's model [45] produces an artificial stiffening effect at the crack tip, even at zero  $\sigma_{xx}$ . Therefore, it appears inapplicable for fracture analysis.

*Does Mohr failure envelope exist?* This question was briefly addressed in the preceding paper [1], in which it was shown that the Mohr circles for the states at  $P_{max}$  of concrete have no envelope. In Fig. 7c,d, the same is now demonstrated for shale, using the results corresponding to the curve of  $G_f$  in Fig. 7c,d.

#### *Microplane model features essential for predicting $G_f$*

The microplane model used in the present simulations is described in detail in [21]. Let us list here briefly the features of microplane model that are advantageous for the modeling of crack-parallel compression effects and distinguish it from the models based on tensors and their invariants.

1) Whereas the microplane model calculates the strain tensor from the stress tensor, the constitutive equation is vectorial, calculating the stress vector from the strain vector on a generic plane of any orientation, called the microplane. The vectors, unlike tensors, can be intuitively related to tensile crack opening, compression splitting or frictional slip.

2) A big advantage is that one can capture the *vertex effect*, e.g., the fact that a shear stress increment applied after compressive stress in the inelastic range has incremental stiffness much softer than elastic (even three-times lower, for concrete), while in all the classical tensorial models (Mohr-Coulomb, Drucker-Prager, von Mises, etc.) a stress increment parallel to the loading surface gives incorrectly an elastic incremental stiffness, even in damage states.

3) Another advantage is that, in M7, the hydrostatic or uniaxial strain compression is always hardening, while the uniaxial compressive stress has a peak and postpeak softening.

4) M7 also delivers correct hysteretic loops under cyclic loading, and reproduces subcritical fatigue crack propagation up to several thousand cycles (in agreement with Paris law).

Early on it was thought that the microplane model was computationally too demanding. For one material point it may run ten-times longer than a tensorial constitutive law. But for a system of millions finite elements, tractable today, the difference in running time is imperceptible because the computational work increases, with the number of displacements, quadratically, but on the constitutive law only linearly.

## **6 LIMITATIONS OF COHESIVE CRACK MODEL AND LEFM, AND CONTRAST WITH CRACK BAND MODEL**

The present results highlight the limitations of the cohesive crack model for quasibrittle materials. In the case of Mode I fracture, this model is defined by a scalar relation between the crack-bridging, or cohesive, normal stress and the relative normal displacement across the crack. In the case of mixed mode fracture,



the model is defined by a relation of crack-bridging normal and shear stresses to the relative normal and shear displacement. But the crack-parallel normal stresses (and strains) cannot be included as the basic force and displacement variables. They can be considered only as parameters affecting the material fracture properties. For the effect of  $\sigma_{xx}$ , [1] gives the formula

$$\frac{G_f}{G_{f0}} = 1 + \frac{a}{1 + b/\xi} - \frac{1 + a + b}{1 + b} \xi^s \quad (1)$$

where  $\xi = \sigma_{xx}/\sigma_{xx,c}$ , and  $\sigma_{xx,c}$  at  $\sigma_{pad} = \sigma_c$ . The curve in Fig. 3A,B is well approximated by  $a = 1.038$ ,  $b = 0.245$ ,  $s = 7.441$  (as shown by dashed curves). These values will, of course, be different for different materials, structure sizes, load histories,  $\sigma_{zz}/\sigma_{xx}$  ratios, etc.

A fully realistic model for quasibrittle fracture must, therefore, well describe the microscale mechanisms mentioned above. This can be either a tensorial, with the FPZ described by a tensorial damage constitutive model with strain softening, coupled with some form of a localization limiter or mesoscale descriptive. The simplest and most widely used model is the crack band model coupled with a physically realistic form of the continuum damage model, here the microplane model, underlying the present simulations.

It should be noted in Fig.6b that there exists a threshold at which the crack-parallel compressive stress  $\sigma_{xx}$  turns from from strengthening to weakening of the fracture resistance. For the present concrete, it is  $\sigma_{xx} \approx -0.75\sigma_c$ . No doubt this threshold varies among different materials, and for some the strengthening phase might not exist. This might be the case for some uniaxial laminate fiber composites, notoriously weak in compression.

From the microplane simulations it is clear that the microcracking in the FPZ produced by high crack-parallel compressive stress  $\sigma_{xx}$  must increase permeability in the FPZ and in its wake as the crack propagates. This is important for various geomechanics problems, and particularly for hydraulic fracturing (or fracking) of gas or oil shale. In [46] it was shown that the permeability and reduced transverse Biot coefficient due to preexisting microcracks with its Biot coefficient is what causes the hydraulic cracks to branch sideways from the wall of a primary hydraulic crack. The present analysis shows that the propagations of the primary crack and secondary branched cracks must be strongly affected by the crack-parallel overburden and tectonic stresses, which are about 80 MPa and 40 MPa at the typical depth of fracking, about 3 km.

Because of their simplicity, scalar models for softening damage have become popular for crack band

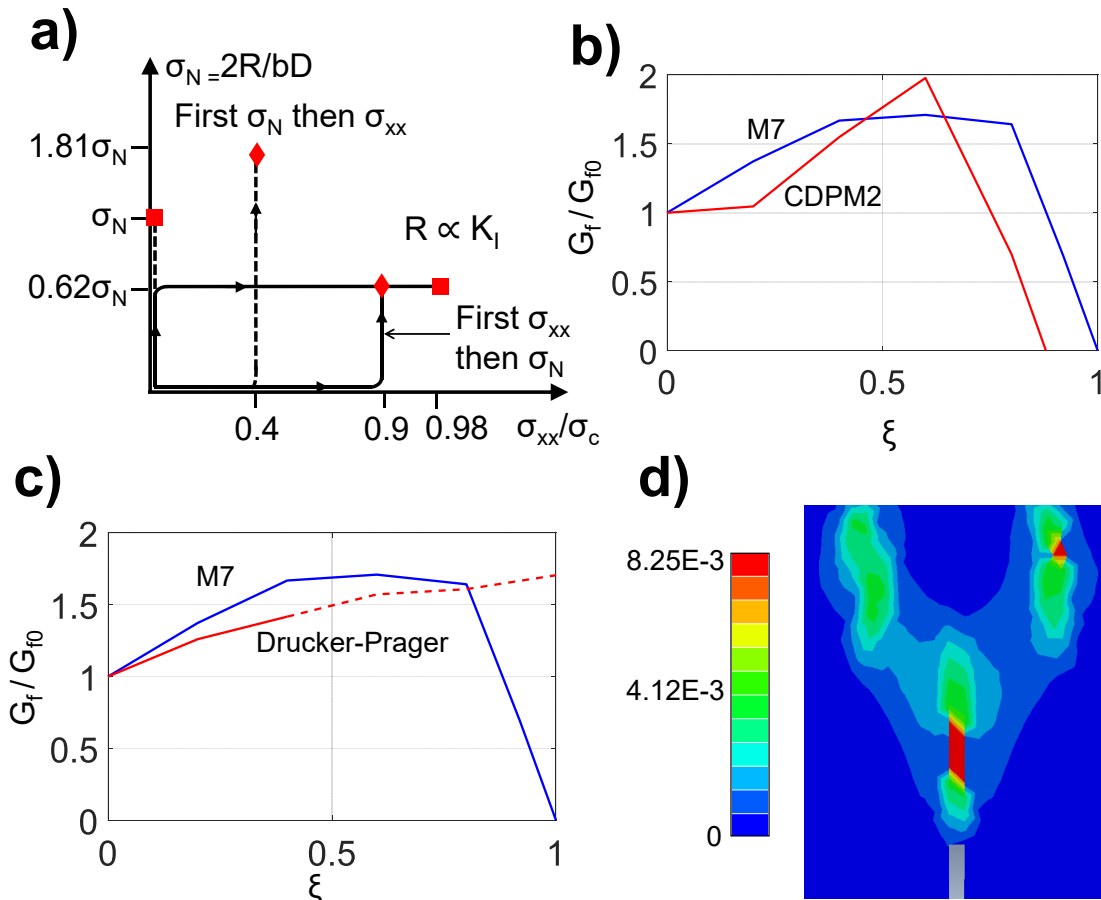


Fig. 9. (a) History dependence of the stress path; (b)  $G_f$  as a function of  $\sigma_{xx}$  predicted by tensorial models by Grassl *et al.* [43]  $\sigma_{xx}$ ; (c)  $G_f$  as a function of  $\sigma_{xx}$  predicted by tensorial Drucker-Prager model [44] a; and (d) A premature failure predicted by D-P ahead of the main crack tip before moment  $\bar{M}$  is applied.

representation of fracture. Some examples are Mazars' isotropic damage model for concrete [47], the nonlocal damage model [48], and all other models using the Kachanov-Hult-Lemaitre type scalar damage measure  $\omega$ . These damage laws may be unrealistic for two reasons—they have not been calibrated by various types of triaxial material tests on specimens of sizes nearly equal to the FPZ (or RVE) size (see 21 types of such tests used to calibrate M7 in [29]). Their use of a single scalar damage parameter,  $\varphi$ , varying between 0 and 1, appears too restrictive (the microplane model and fracture characteristics has several independent damage parameters). All these models are usable only if the crack-parallel normal stress is *a priori* known to be negligible (or below about 10% of compressive strength) in all the finite elements within the damage zone. As for peridynamics, no comments are necessary [49].

The fracture models characterized only by  $G_f$  can be used if reprogrammed to vary  $G_f$  as a function of the crack-parallel stresses,  $\sigma_{xx}$  and  $\sigma_{zz}$ . This, of course, ignores history effects, whose severity has not yet been clarified. As for the cohesive crack model, its softening stress-displacement law would have to be varied depending on  $\sigma_{xx}$  and  $\sigma_{zz}$ . But this brings up the questions of whether the cohesive softening curve should be scaled as a whole (which would scale both  $G_F$  and  $G_f$ , or only in its initial part (controlled by  $G_f$  only), what should be the proportion of such scalings, and whether a horizontal scaling is also necessary. Again, this would miss history effects.

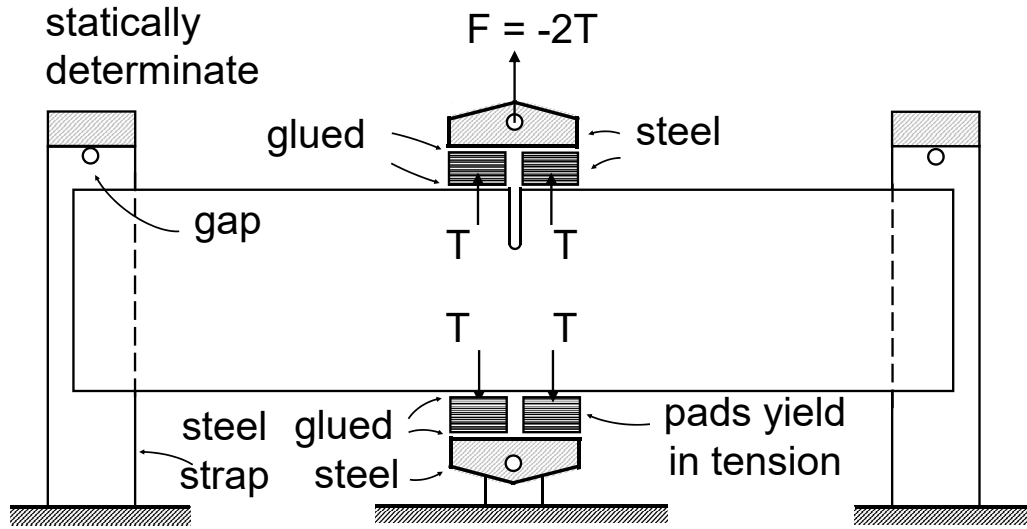
## **7 GAP TEST FOR CRACK-PARALLEL TENSION**

Finally, it may be pointed out that the gap test may be easily adapted to measure the effect of crack-parallel tension on  $G_f$ . The setup shown in Fig. 10 is self-explanatory.

## **8 CONCLUSIONS**

1. Moderate crack-parallel compressive stress, in-plane or out-of-plane, drastically increases, even doubles, the Mode I fracture energy  $G_f$  (or fracture toughness  $K_{Ic}$ ) of concrete, and probably also shale and various rocks, coarse-grained ceramics and sea ice. This can be explained by an increase of hydrostatic pressure raising friction on inclined planes.
2. High crack-parallel compressive stress close to the compression strength limit drastically decreases  $G_f$  and eventually reduces it to zero. This can be explained by frictional slip on inclined planes and splitting microcracks, causing lateral dilation.
3. The cohesive crack model (CCM), as a line crack model with a scalar relation between crack-bridging stress and relative displacement of crack faces, cannot capture these phenomena, since the crack-parallel strains are not the thermodynamic variables in this model. So  $\sigma_{xx}$  and  $\sigma_{zz}$  must be used as parameters which, however, appears to be only a crude approximation which cannot capture the effect of the triaxial stress and strain history. The same objection also applies to the recent cohesive crack model that is enriched with crack-parallel strain by shrinking a crack band into a line [50].
4. To capture the experimentally evidenced effect of crack-parallel compression in general, either a microscale descriptive model or a tensorial constitutive model for softening damage must be used. This is effectively done with the crack band model coupled with the microplane damage constitutive law. This law can mimic the effects of inelastic frictional slips and microcrack formation on planes of different orientations. The damage laws in the phase-field models restricted to only a single damage parameter

### STAGE 1



### STAGE 2

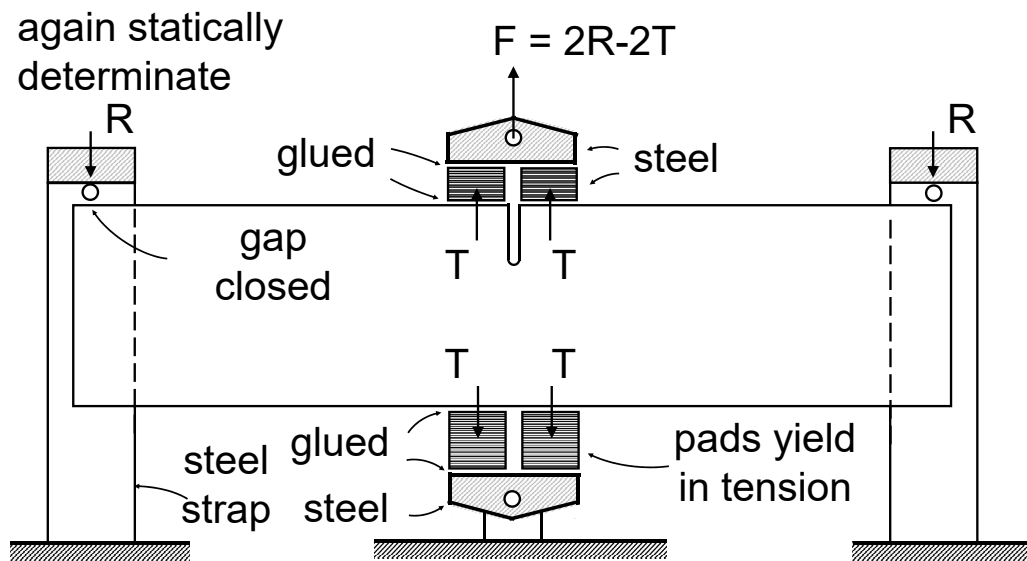


Fig. 10. Gap test for crack-parallel tension.

do not appear to be realistic.

5. Linear elastic fracture mechanics cannot capture the effect of crack-parallel compression. As a line crack model, CCM can neither. Nor can the computational models based on LEFM or CCM.
6. An effective method for fracture testing with crack-parallel compression is the gap test, in which the compression is generated by plastic pads capable of perfectly plastic yielding, and the supports are installed with a gap that closes only after the pads begin yielding. The gap test is possible for both compressive and tensile crack-parallel stress.

7. The results are of particular interest for: a) the shear failure of reinforced concrete beams and punching of slabs; b) fracture of prestressed concrete, c) for hydraulic fracturing of shale, at which the overburden and tectonic stress introduce significant crack-parallel compression; c) fiber composites, where buckling of embedded fibers reduces compression strength; d) propagation of the front of slip on earthquake fault, subjected to enormous crack-parallel tectonic and overburden stresses; e) splitting fracture of sea ice plates pushing against fixed objects, or in front of icebreaker; f) burst of very deep boreholes and mine stopes; g) high biaxial tensile stresses in composite aircraft fuselage; h) cyclic and static fatigue crack growth under crack-parallel in-plane and out-of-plane stresses; etc.
8. The Mohr circles for various levels of crack parallel stress in shale do not have an envelope. Therefore, general use of Mohr-Coulomb failure criterion for shale or other quasibrittle materials is unwarranted.

*Note:* A powerful alternative to the present analysis of concrete is the Lattice Discrete Particle Model (LDPM) [51,52] mimicking the heterogeneous microstructure and mesoscale interactions. It will be pursued in a forthcoming paper (also it may be remarked that making more than 27 experiments was prevented by the current virus pandemic).

#### **Appendix A. Alternative test methods and related previous studies**

As an alternative to the present test, a system of two balance beams shown in Fig. 11, was fabricated and tried to provide reactions at both beam ends and notch corners, the ratio of these reactions being controlled by moving the locations of the supports at base. But it proved tricky to keep the system stable and properly aligned before applying the load. Another bigger problem was seen in the evaluation. This test would have to be conducted for several different specimen sizes  $D$  and heavy I-sections, which must provide a much larger stiffness than the largest specimen size. The self-weight of these I-sections would add up to  $\sigma_{xx}$  or bending moment non-proportionately, which will create complication to the fracture analysis. The biggest problem is that the compressive force would not be constant but (in theory) raised proportionally that would make it difficult to deduce the effect of  $\sigma_{xx}$  on  $G_f$ . Since  $\sigma_{xx}$  does not matter for the size effect method, this method could still be used, but it would deliver a fracture energy corresponding to a certain history of  $\sigma_{xx}$  rather than to any specific value of  $G_f$ . Complex optimization of tests with many different histories, using an assumed constitutive damage law, would be required, to obtain a rather uncertain result for the effect of  $\sigma_{xx}$  on  $G_f$ . The constancy of  $\sigma_{xx}$  is what makes the gap test simple and unambiguous.

Additional hydraulic jacks were used in 1995 by Tschegg et al. [53] to introduce crack-parallel compression in an elaborate modification of the wedge-splitting test, in which separate hydraulic jacks were used to

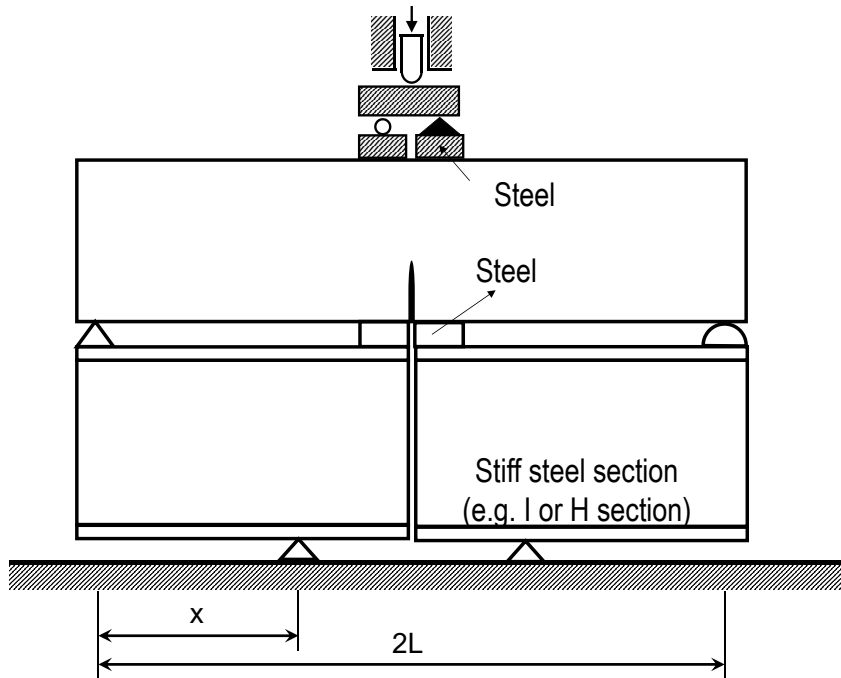


Fig. 11. An alternative test method considered to generate  $\sigma_{xx}$  with proportional loading.

introduce crack-parallel compression. The results confirmed the hint from the 1987 microplane model that crack-parallel compression should have some effect but no clear conclusion could be drawn. However, the evaluation was aimed at  $G_F$  rather than  $G_f$ , and thus was compromised by the shape, unknown at that time, of the complete softening law with its long tail (as in Fig. 1). The evaluation also suffered from the complexity of stress field in which the maximum compression occurs away from the notch tip, and also from the effects of bending moment due to the weight of clamping frames and hydraulic jacks, as well as from the friction under the jacks. The main problem, unknown in the 1990s, was that  $G_F$  evaluation by the work-of-fracture method gives rather ambiguous results unless scaled tests are performed for several sufficiently different sizes [25]. Another serious problem is the energy dissipation rate (or the J-integral) is markedly reduced during the initial formation of the FPZ near the notch tip and during the terminal phase on approach to the opposite phase (Fig. Y).

If the FPZ is not small compared to the ligament cross section, which is always the case for concrete, the work-of-fracture per unit crack length severely underestimates the steady -state energy dissipation and causes a major error.

As another alternative, one might also think of imposing a compressive displacement with pair of stiff clamping frames mounted on the beam. But the stress induced would relax (by about 30%), due to concrete creep and shrinkage during the test. To get  $G_f$  as a function of strain  $\epsilon_{xx}$  instead of  $\sigma_{xx}$ , the clamping

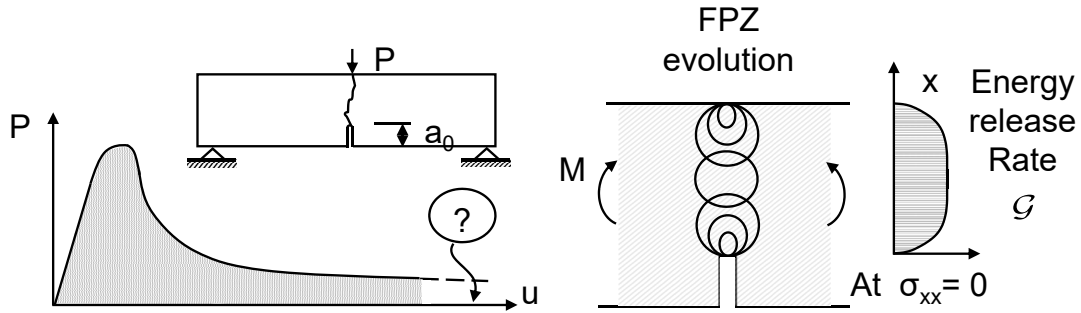


Fig. 12. Load deflection curve in work-of-fracture test, with a long tail hard to capture completely (left), and growth of FPZ at the start of crack propagation from the notch, and decrease at the end (right).

frames would have to be very stiff and heavy, to maintain constant  $\epsilon_{xx}$ , and would have different relative weights for different specimen sizes. Another possibility would be hydraulic flat jacks within these frames, similar to those of Tschegg but equipped with computer control. Interpretation would anyway be fraught with ambiguity.

For metals assumed to follow the Ramberg-Osgood  $J_2$  power-law hardening plasticity, Hutchinson, Tvergaard, Shih, O'Dowd, Betegón and Hancock [54–57]) showed analytically, in the early 1990s, that the in-plane crack-parallel stress (called the T-stress), and generally the stress triaxiality, affect the J-integral and the singular plastic-hardening HRR field surrounding a sharp crack tip. The combined effect of  $\sigma_{zz}$  and  $\sigma_{xx}$ , and plane strain versus plane stress, seem not to have been studied. But, as commented on in more detail in [1], these are different problems. The present analysis does not apply to metals (with the probable exception of the micrometer scale). Furthermore, Cotterell and Rice [58] showed analytically that a high T-stress may cause the LEFM crack path to deflect sideways, but this, too, is a different problem.

### Appendix B. Concrete characterization and further gap-test details

To minimize the scatter of mechanical properties, all the specimens were cast within a few hours from the same batch of concrete delivered by a ready-mix supplier (Ozinga co.). Normal concrete of specific compression strength  $f'_c = 27.58$  MPa (4000 psi) at 28 days of age was used (data on separate tests made by Ozinga are reported in Appendix A). The slump range was 7.62–12.70 cm (3.00 – 5.00 in.), and the specific air content 0–3%. In the mix, one cubic yard contained 470 lb. of cement (ASTM C150, 100 lb. of blast-furnace slag (labeled C989), 1750 lb. of coarse aggregate (ASTM C33#67), 1500 lb. of fine aggregate (ASTM C33#2); water-cement ratio  $w/c = 0.45$ .

The specimens for material characterization and model calibration (Fig. 1) include: cylinders for compression tests of diameter 101.6 mm and length 203.2 mm; square prisms for compression tests of side

76.2mm and length 152.4 mm; cylinders for splitting tests of diameter 101.6 mm and length 203.2 mm; and prisms for splitting tests of side 76.2 and length 152.4 mm. The beams for fracture tests, geometrically scaled, were of three sizes; small: 101.6 × 101.6 × 406.4mm; medium: 101.6 × 203.2 × 812.8 mm; and large: 101.6 × 203.2 × 812.8 mm. The notch/depth and span/depth ratios were  $a/D = 0.3$  and  $2L/D = 3.75$ , respectively. For tests with various crack-parallel pressures, the beam dimensions were 76.2 × 101.6 × 406.4 (all dimensions were in mm). The elasto-plastic loading pads had sides  $S$  of ratio  $S/D = 1/4$  (Fig. 1e). All the specimens were test within 3 weeks, but the effect if this age difference on the degree of hydration and strength was negligible since the specimens were 1 year old. The specimens were all cured in a fog room until the time of test.

### **Appendix C. Material calibration experiments**

Aside from the results of the gap tests for different specimen sizes at zero  $\sigma_{xx}$ , the properties of concrete were calibrated by tests of uniaxial compression of cylinders and prisms Fig. 13a,b. These calibrations are sufficient for M7 to characterize the behavior of normal concrete, which was validated by Brazilian splitting tests on both cylinders and prisms (see Fig. 13c, where the error bars indicate one standard deviation based on 4 tests for each case). The same model was utilized to predict the results of the gap test at non-zero  $\sigma_{xx}$ . For uniaxial compression, the specimens were capped with sulfur to make sure the loaded surface would be flat and almost frictionless. Linear Variable Differential Transformers (or LVDT) were used to record two crack-parallel displacements, as well as one lateral displacements used to control the loading, see Figs. 1a,b (ASTM C39). In the Brazilian tests (Figs. 1c,d), the loading strip was 12.7 mm wide and the samples were held in place at the beginning by a mild pressure (ASTM C496). The uniaxial tests of specimens of both types were conducted in the Tinius Alson frame with maximum load of 1000 kips, and the Brazilian split experiments the MTS loading frame of capacity 50 kips, both at the loading rate of 0.00635 mm/sec. Every test took for approximately  $\approx 0.5$  hour.

Figure. 13 shows the results on uniaxial compression strength of cylindrical and prismatic specimens (Figs. 3a,b) and their corresponding Brazilian splitting tensile strength (Fig. 3c). In Fig. 3a,b, the dashed curve shows the upper and lower envelope of data from three tests, whose average is represented by the dotted curve. This result is consistent with the average strength  $f'_c = 42.74$  MPa reported by concrete supplier. Optimum fits of these results were obtained with the microplane constitutive model M7 [21, 29] shown by solid curves. The M7 was implemented in Abaqus using the crack band model with element size 12 mm. The dotted and solid curves are consistent and serve to indicate the level of compression on



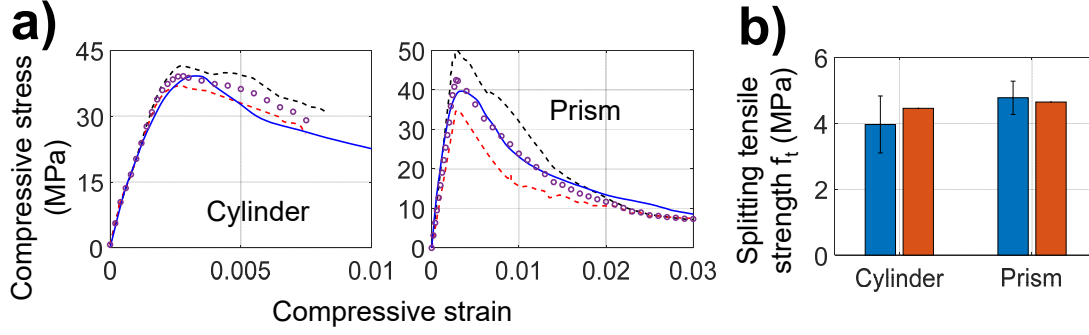


Fig. 13. Calibrations and validations to obtain parameters for M7-CBM: (a) Uniaxial compression of cylinders; (b) Uniaxial compression of square prisms; and (c) Brazilian splitting tests.

notched beams.

#### Appendix D. Compression stiffness of plastic pads and their optimization

For simplicity, consider a circular pad with radius  $r$  and polymer layer thickness  $h$  (Fig. 13). In the plastic regime, we assume the polymer is incompressible and has a very small but non-zero shear modulus  $\mu$  (if  $\mu$  were zero, the plasticized polymer would flow out). The boundaries of the layer are confined and behave as rigid. Volume conservation in the disc of radius  $x \leq r$  requires that  $\pi x^2 w - 2\pi s h u$  where  $w$  - relative displacement of decrease of thickness and  $u$  - average increase of radius  $x$ . Hence  $u w x / 2h$ . Given that the polymer does not slide on the layer surfaces, we may assume the radial displacement profile across layer thickness to be parabolic Fig. 12). In that case, the maximum displacement is  $1.5\times$  higher than  $u$ , the mean, and thus the profile of shear angle is  $ga = \partial u / \partial y = 3wxy / h^3$ , where  $y$  is the transverse coordinate. The strain energy density with respect to radial coordinate  $x$  then is  $\bar{W}(x) = 2 \int_0^{h/2} \mu \gamma^2 / 2 dy = 3\mu w^2 x^2 / 8h^3$ . Conservation of energy requires the work of compression of the layer by  $w$  to equal the strain energy increment of the whole layer, i.e.,  $W = Hw^2 / 2 = \int_0^r \bar{W} \cdot 2\pi dx$  where  $H$  = compression stiffness of the layer. Integration yields:

$$H = k_p \mu A^2 / h^3, \quad k_p = 3/8\pi \quad (2)$$

where  $A = \pi r^2$  = area of the pad. this equation may also be used as an approximation for a square pad of the same area (an accurate solution is more complicated). The pads should be shaped so as to minimize  $H$  (subject to Haringx's critical load formula [59] for shear buckling governing the elastomeric bearings of bridges), or minimizing  $A$ , which means replacing one pad with several small ones (or drilling big holes).

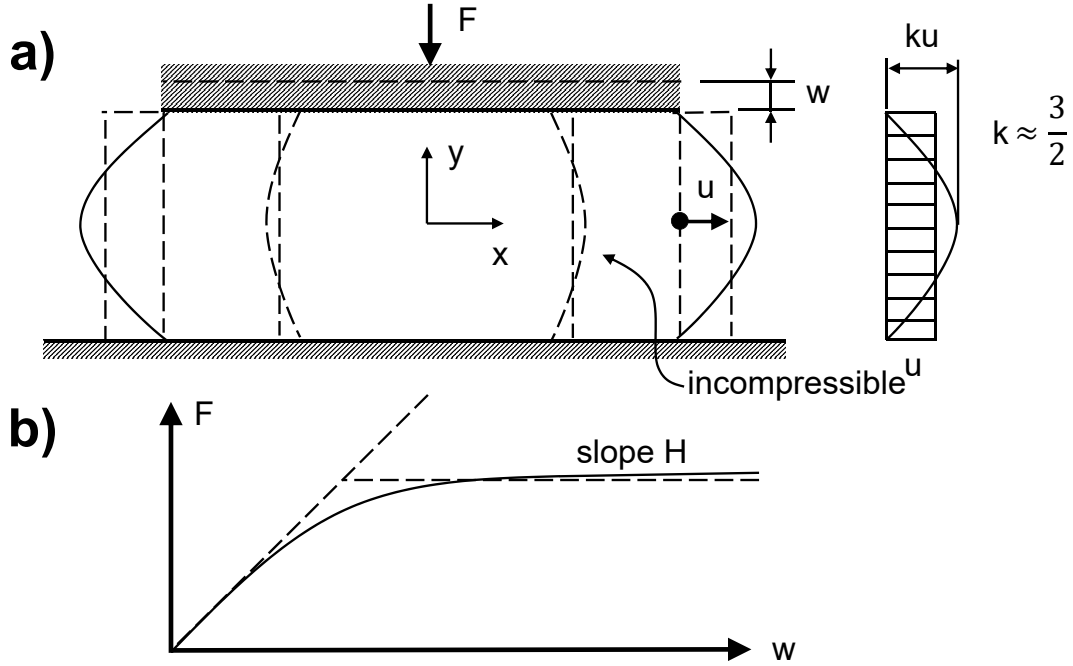


Fig. 14. (a) A circular plastic pad under compressive force; and (b) The shape of the pad should be designed to minimize  $H$ .

Haringnx's formula gives the critical ratio  $\lambda_{cr} = \sqrt{A}/h$  for buckling.

Second, let us analyze an elongated  $l \times L$  rectangle (the pad in Fig. 14), with  $L \gg l$ , approximately in 1D, coordinate  $x$ . Due to incompressibility, the axial strains are distributed as  $u = (w/h)x$ , and the strains are  $\gamma = (6u/h^3)y$ . The work per unit length in  $x$  is  $\bar{W} = 2 \int_0^{h/2} \frac{1}{2} \mu \gamma^2 dy = 3\mu w^2 x^2 / 2h^3$ . By integration,  $W = 2b \int_0^{l/2} \bar{W} dx = b\mu w^2 l^3 / 8h^3$ . Setting  $W = \frac{1}{2} H w^2$ , we get

$$H/L = (\mu/4)(l/h)^3 \quad (3)$$

The same formula approximately applies to an annular pad with internal and external radii  $r_1$  and  $r_2$  provided that  $l = r_2 - r_1 < r_1/4$ . It also roughly applies to a pad with many sufficiently big regularly spaced holes (Fig. 14) of radius  $r$  provided that approximately  $\sqrt{A/A_s} - 1 < 1/4$  where  $A$  = total area and  $A_s$  = area of the holes. If  $> 1/4$ , one can interpolate. Other polymers, e.g. polyethylene, PDMS, can be also used to get a different  $\mu$ . A possible alternative to the plastic pads are blocks of perfectly plastic metal, e.g., tin (Sn).

## Appendix E. Size effect method to identify $G_f$ and $c_f$

To determine  $G_f$ , one needs the LEFM stress intensity factor,  $K_I$  [14, 26]. Although the load configuration is close to 3-point bending, four-point-bending with a small but finite distance between the loads gives better accuracy (Fig. 3c). According to [60]:

$$K_I = \sigma_N \sqrt{D} k(\alpha) \quad \text{where } \sigma_N = P/bD \quad (\alpha = a/D) \quad (4)$$

$$k(\alpha) = \frac{1.1682(2l-s)\sqrt{\pi\alpha}}{8\beta^{3/2}} \left( 5 - 10\alpha/3 + \alpha^2 + 40\alpha^2(1-\alpha)^6 + 3e^{-6.134\alpha/(1-\alpha)} \right) \quad (5)$$

where  $P$  = total load (sum of two loads) applied at center-span,  $\sigma_N$  = nominal strength of beam;  $2L$  = span between the supports,  $S$  = distance between the two center-span loads;  $l = L/D$ ,  $s = S/D$ ;  $b, D$  = thickness and depth of beam;  $a$  = crack (or notch) length,  $\alpha = a/D$  = relative crack length;  $\sigma_N$  = nominal bending strength of each size;  $f(\alpha)$  = dimensionless stress intensity factor. The size effect law (SEL) reads:

$$\sigma_N = Bf_t(1 + D/D_0)^{-1/2} \quad (6)$$

$$1/\sigma_N^2 = (1/B^2 f_t^2 + D/B^2 f_t^2 D_0) \Rightarrow Y = C + AD \quad (7)$$

For data fitting, the SEL may be converted to a linear regression plot of  $1/\sigma_N^2$  versus  $D$  as in 7 [14, 61]. To determine the size effect parameters, we need not only  $k(\alpha)$  but also its derivative [14, 26]:

$$\begin{aligned} k'(\alpha) = \frac{dk(\alpha)}{d\alpha} = & \frac{0.8411(2l-s)\sqrt{\pi}}{8(1-\alpha)^{3/2}\sqrt{\alpha}} \left( 5 - 10\alpha/3 + \alpha^2 + 40\alpha^2(1-\alpha)^6 + 3e^{-6.134\alpha/(1-\alpha)} \right) \\ & + \frac{2.523(2l-s)\sqrt{\pi\alpha}}{8(1-\alpha)^{5/2}} \left( 5 - 10\alpha/3 + \alpha^2 + 40\alpha^2\beta^6 + 3e^{-6.1342\alpha/(1-\alpha)} \right) \\ & + \frac{1.6823(2l-s)\sqrt{\pi\alpha}}{12(1-\alpha)^{3/2}} (-5 + 3\alpha + 120\alpha((1-\alpha)^6 - 360\alpha^2(1-\alpha)^5 \\ & - 27.6(1/(1-\alpha) + \alpha/(1-\alpha)^2) e^{-6.134\alpha/\beta}) \end{aligned} \quad (8)$$

Upon rearranging Eq. (6) as linear regression and fitting maximum load data, Eq. (7), the fracture energy and the characteristic material length (roughly 40% of actual FPZ length) can then be computed from [14,26]:

$$G_f = K_f^2/E, \quad K_f = Bf_t\sqrt{D_0}k(\alpha_0) \quad (9)$$

$$c_f = \frac{D_0k(\alpha_0)}{2k'(\alpha_0)} \quad (10)$$

Note that the crack-parallel compression is not a parameter in LEFM, and so, in LEFM, it cannot affect function  $k(\alpha)$  and the equations relating  $G_f$  and  $c_f$  to  $k(\alpha)$ .

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